Money and banking in a New Keynesian model*

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Abstract

This paper studies a New Keynesian model with a banking system. The central bank targets the interest rate on short safe bonds that are held by banks to back inside money and hence earn convenience yield for their safety or liquidity. Central bank operating procedures matter. In a floor system, the reserve rate and the quantity of reserves are independent policy tools that affect banks' cost of safety. In a corridor system, increasing the interbank rate by making reserves scarce increases banks' cost of liquidity and generates strong pass-through to other rates of return, output and inflation. In either system, policy rules that do not respond aggressively to inflation – such as an interest rate peg – need not lead to self-fulfilling fluctuations. The stabilizing effect from an endogenous convenience yield is stronger when there are more nominal rigidities in bank balance sheets.

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1 Introduction

This paper is motivated by two familiar facts on money and banking. The first is the "short rate disconnect": interest rates on short safe bonds targeted by central banks are not well accounted for by asset pricing models that fit expected returns on other assets such as long terms bonds or stocks. A related fact is that short safe bonds that earn policy interest rates, such as overnight interbank loans and central bank reserves, are predominantly held by intermediaries.[1] The second fact is the existence of a stable and relative inelastic money demand schedule for appropriately defined broad money balances (for example, Lucas and Nicolini (2015), Benati et al. 2021). In particular, estimated elasticities of money demand are far below conventional numbers for the intertemporal elasticity of substitution (IES).

The textbook New Keynesian model is not consistent with both facts on bonds and money. On the one hand, it identifies the policy rate with households' rate of return on savings, or the short rate in households' stochastic discount factor. In other words, it imposes perfect pass-through from the policy rate to all other rates in the economy, thus giving the central bank a powerful lever to affect intertemporal decisions. On the other hand, microfoundations for the textbook model assume that either (i) the economy operates at a "cashless limit", so the model does not speak to the role of the quantity of money or (ii) utility is separable in consumption and real balances, which implies an elasticity of money demand as high as the IES.

This paper studies a New Keynesian model with a banking system that features both a short rate disconnect and a stable, inelastic money demand schedule. To capture the role of money as a medium of exchange, real inside money balances created by banks enter utility as a complement to consumption. A short rate disconnect arises because short safe bonds are held by banks to back inside money – the convenience yield on those bonds reflect their benefit as safe collateral. We show how in such a world the "plumbing" of the economy – the nature of payment flows – as well as the structure and assets of the banking system matter for the transmission of monetary policy. Moreover, the precise operating procedures of the central bank – such as whether it adopts a corridor or a floor system – are important for what policy tools are available and how policy can avoid self-fulfilling fluctuations.

According to our model, the standard New Keynesian setup approximates policy transmission via banks fairly well when the central bank runs a corridor system with a fixed interest rate on reserves, that is, it supplies reserves elastically to hit a target for the interbank loan rate, as the Fed did prior to the 2008 financial crisis. With this operating procedure, policy

[1]The short rate disconnect has been a stylized fact in the empirical literature on the term structure of interest rates since Duffee (1996). Lenel, Piazzesi and Schneider (2019) provide evidence of its connection to bank balance sheets.

works mostly through banks' cost of liquidity, defined as the spread between the interbank rate and the reserve rate, both set by the central bank. The quantity of reserves is adjusted by the central bank trading desk to implement its desired spread; it plays no independent role as a policy instrument. While the short rate disconnect implies that pass-through from the policy rate to the interest rate on savings is imperfect, it is relatively strong at typical parameter values, because banks' supply of inside money is sensitive to their cost of liquidity.

When the central bank varies the interest rate on reserves, however, the standard model does not provide a good abstraction for how monetary policy works, for two reasons. First, raising interest rates does not require a decline in the quantity of reserves, which instead serves as an independent policy instrument. In particular, the central bank in our model can run a floor system with ample reserves, as many central banks have done over the last decade. Here the central bank supplies a quantity of reserves that is much larger than what banks require for liquidity management. The interbank loan market then shuts down and there is a single short bond rate, set by the central bank. Banks value reserves only as collateral to back inside money, not for their liquidity. A reduction of reserves, for example through unwinding an asset purchase program, is contractionary because it lowers the average quality of bank collateral, even if interest rate policy does not change.

A second key difference between a floor system and the standard model is that interest rate policy in a floor system no longer works through banks' cost of liquidity – which is constant at zero – but instead through banks' cost of safe collateral, measured by the spread between the interest rate on savings and the interest rate on reserves. A higher policy rate therefore does not make liquidity more expensive, but instead makes safe collateral cheaper, which lowers banks' cost of providing money. The difference between these alternative transmission mechanisms is particularly pronounced when money and consumption are complements. In the standard model, a higher cost of producing money feeds through to the cost of producing output thereby generates a strong contractionary effect of monetary tightening on output. In our banking model, cheaper collateral dampens this effect, so interest rate policy is much less powerful.

As an alternative setup without banks that approximates policy transmission with variable interest on reserves, we suggest a simple tweak to the standard model: equate the policy rate with the interest rate *on money* rather than the interest rate on savings. In other words, we consider a hypothetical world where the government offers interest-bearing central bank digital currency (CBDC). We show that this *CBDC model* captures the key elements of our banking model with variable interest on reserves. The simplification is that interest rate policy directly affects households' cost of liquidity, rather than indirectly through the convenience yield on short safe bonds that back money. The CBDC model also features two policy instruments, and dampening of interest policy relative to the standard model. It becomes closer to the standard model as the elasticity of money demand increases. [2]

Our results follow from a familiar set of assumptions on the role of money and banking in the economy. First, inside money issued by banks earns a convenience yield for its liquidity, measured by the spread between the interest rate on savings and the interest rate on money: households' cost of liquidity. Second, banks face leverage constraints, because inside money must be backed by collateral. Importantly, both households' cost of liquidity and banks' cost of safety - spread between the rate on savings and the rate on reserves – is always positive, even in a floor systems when banks' cost of liquidity is zero. Third, inside money is liquid so heterogeneous banks are subject to sudden in- and outflows of money as they process customers' payment instructions. Finally, pass-through from the policy rate to other rates occurs because *total* risk-adjusted expected returns – pecuniary expected returns plus convenience yields – on all assets are equated in equilibrium.

We characterize the macro dynamics of our model by a difference equation that is a simple extension of the familiar three New Keynesian equations. Behavior of the nonbank private sector is summarized by a New Keynesian Phillips curve as well as an intertemporal Euler equation. Since we allow for complementarity between money and consumption, the cost of production reflects in part households' cost of liquidity, as in models of the cost channel of monetary policy. A third equation is a standard money demand relationship that relates real balances to households' cost of liquidity. Two additional equations summarize aggregate prices and quantities of bonds and money that are consistent with (partial) equilibrium in fixed income markets: banks price households' cost of liquidity at a constant markup over marginal cost, and supply a quantity constrained by available collateral. The plumbing as well as policy shapes the parameters of these equations and hence the transmission mechanism.

To see how endogenous convenience yields affect the transmission of interest rate policy under our assumptions, suppose the central bank raises the interest rate on short safe bonds held by banks: it raises the target for the interbank rate in a corridor system, or the interest rate on reserves in a floor system. Standard New Keynesian logic says that nominal rigidities imply a higher real short rate and lower nominal spending. However, lower nominal spending lowers the convenience yield on inside money and hence on short safe bonds that back inside money, be they interbank loans or reserves. The overall return on safe short bonds therefore does not increase as much as the policy rate itself. Since pass-through to other interest rates occurs to

[2] The logic of the CBDC model is relevant not only when the central bank runs a floor system, but whenever policy moves the interest rate on reserves without changing banks' cost of liquidity. For example, a system with a fixed size corridor such that the reserve rate moves in lockstep with the target for the interbank rate, so the cost of liquidity is constant at a positive spread, behaves similarly to a floor system.

equate *total* risk adjusted returns, the response of the convenience yield to spending dampens the policy impact on output and inflation. The effect works through the value of bonds as collateral–banks' cost of safety–and is thus present in both floor and corridor systems.

In contrast to the standard model, our model says that interest rate rules that do not aggressively respond to inflation need not make the economy susceptible to self-fulfilling fluctuations. Consider for example an interest rate peg. Can there be a self-fulfilling recession? If agents believe that output is temporarily low, inflation slows as firms anticipate lower cost. With a pegged nominal rate, the real rate increases. In the standard model, the expected real return on all assets increases: lower demand makes the recessionary belief self-fulfilling. In our model, in contrast, lower spending lowers the convenience yield, which in turn keeps the expected real return on other assets low. Put differently, the Taylor principle – lower inflation should lead to a lower real interest rate – can hold for the interest rate on savings, even if it does not hold for the policy rate of the central bank. Endogenous adjustment of the convenience yield substitutes for policy as a stabilizing force.

Our paper adds to a growing literature on New Keynesian models with financial frictions, dating back to Bernanke, Gertler and Gilchrist (1999). Recent work has focused on financial frictions in the banking system; see for example Cúrdia and Woodford (2010), Gertler and Karadi (2011), Gertler et al. (2012), Christiano, Motto and Rostagno 2012, Ireland (2014), Del Negro et al. (2017), Brunnermeier and Koby (2018) or Wang (2019). In these models, banking also matters for transmission and there can be imperfect pass-through from the policy rate to deposit or loan rates. The papers nevertheless share the feature of the standard model that there is direct pass-through from the policy rate to the short rate, and therefore households' nominal stochastic discount factor. They do not speak to the short rate disconnect, the key fact that motivates our analysis.[3]

Diba and Loisel (2019) study the determinacy properties of a New Keynesian model with banks at the zero lower bound. In their setup, reserves are an input into bank lending, and the government commits to a nominal path of reserves. They establish local determinacy under the assumption that reserves remain scarce at the zero lower bound. In our model, in contrast, determinacy properties follow from the convenience yield of bank liabilities. It is not essential that reserves are scarce, that the government commits to a nominal path of reserves or that the policy rate is the reserve rate. In fact, our comparison of operating procedures focuses on

[3]Much recent work on New Keynesian models has been motivated by the zero lower bound on interest rates, and various "puzzles" such as large fiscal multipliers or strong impact of forward guidance. In this paper, we do not focus on a lower bound. Instead, our goal is to extend the New Keynesian model in a way that is consistent with data on interest rates as well as holdings of short safe bonds. From this perspective, 2008 is a watershed because the Fed adopted a floor system that made liquidity cheap for banks. That decision is still relevant now that the level of interest rates has risen again.

times away from the zero lower bound when either (i) reserves are scarce and the central bank targets an interbank rate – the US policy regime before the financial crisis – or (ii) reserves are abundant and the central bank sets the reserve rate – the US regime after the crisis.

There is recent work on New Keynesian models with convenience yields on other assets. In particular, Hagedorn (2018) studies a HANK model with uninsurable income risk and a riskfree asset. Some consumers are not on their intertemporal Euler equation, so that their marginal rate of substitution is not equated to the interest rate. Michaillat and Saez (2018) assume that wealth is a separate argument in utility, in addition to consumption. In both cases, a convenience yield is priced into assets that serve as a store of value for households. Our perspective here is different: we emphasize the convenience yield on assets held by banks that drives a wedge between the policy rate and the rate at which households save, as we see in the data. Our mechanism is thus complementary to the above effects. For example, a HANK model with banks might feature weak pass-through from the policy rate to the rate on household savings.

More generally, our model builds on a long tradition of asset pricing with investors who face liquidity or collateral constraints, dating back at least to Lucas (1990), Kiyotaki and Moore (1997) and Geanakoplos (2003). Recent work has emphasized the role of constrained intermediaries, see for example Brunnermeier and Pedersen (2009), He and Krishnamurthy (2013) or Bocola (2016). Our model also features "intermediary asset pricing" but differs from much of the literature in that banks are firms that maximize shareholder value and can costlessly recapitalize. The mechanism we emphasize does not require frictions in equity markets, and does not rely on financial accelerator dynamics.

A convenience yield on short bonds is often captured by making bonds an argument in utility, see for example Bansal and Coleman (1996), Krishnamurthy and Vissing-Jorgensen (2012) or Nagel (2016). Lenel, Piazzesi and Schneider (2019) take a closer look at the quantitative asset pricing implications of the approach we follow here. They show that bank optimization implies an observable pricing kernel based on bank balance sheet ratios that accounts well for the short rate disconnect, especially at business cycle frequencies.

We also build on a growing literature that studies macroeconomic effects of the structure of the banking system. In particular, several authors have emphasized the importance of market power in deposits markets; see for example Yankov (2014), Driscoll and Judson (2013), Duffie and Krishnamurthy (1996), Egan, Hortacsu and Matvos (2017), Drechsler et al. (2018) or Corbae and D'Erasmo (2013). In addition, there has been recent interest in bank liquidity management, for example Bianchi and Bigio (2021), De Fiore, Hoerova and Uhlig (2018), or Piazzesi and Schneider (2018). Both features matter for the quantitative relevance of our mechanism; our results suggest that studying them further is important for understanding the transmission of monetary policy.

A key feature of our model is the distinction between several payment instruments and their potential scarcity, in our case, reserves and deposits. The link between scarcity of payment instruments and convenience yields is well established in monetary theory. In particular, Kiyotaki and Moore (2005) and Venkateswaran and Wright (2014) have shown how assets that back payment instruments can inherit their convenience yields, an effect that is also central to our mechanism. The literature has typically studied the coexistence of multiple payments used by households, for example currency and various types of deposits; see also Rocheteau, Wright and Xiao (2018), Andolfatto and Williamson (2015), Lucas and Nicolini (2015), Benigno and Nistico (2017) and Ennis (2018)). We abstract from currency and emphasize instead a layered payment system in which households only pay with inside money, and only banks pay with outside money directly issued by the government.

The paper is structured as follows. Section 2 presents the simple model of central bank digital currency to introduce the key effects. Section 3 studies a partial equilibrium model of banks provision of liquidity and the pricing of fixed income claims that serves as a module for the full macro model with banks in Section 4. Proofs and derivations are collected in the Appendix.

2 Monetary policy with a convenience yield: a minimal model

In this section we study a minimal model of a central bank targeting an instrument with a convenience yield: money earns a convenience yield because it enters the utility function. Households and firms solve the same problems as in textbook treatments of the New Keynesian model. The only difference is that the central bank sets the quantity as well as the interest rate on money, as opposed to the short rate of the representative agent's stochastic discount factor. A special case of the setup is thus a New Keynesian model with a money growth rule. The model is more general, however, because it explores a larger set of rules for both interest rates and the money supply.

Our interpretation is that there is a central bank digital currency (CBDC): everyone has deposit accounts at the central bank, which controls both the nominal quantity and the interest rate. The short rate, like nominal rates of return on all other assets, adjusts to clear markets. Our interest in this model stems from its formal similarity to the banking models in Sections 3 and 4. We will show that the same mechanisms are at work both when the central bank makes reserves abundant – hence controlling their price and quantity – and when the central bank elastically supplies reserves to hit a fed funds rate target. Details of the banking system can be understood as altering the coefficients of policy rules in the model of this section.

2.1 Setup

Every period, the representative household chooses consumption goods C_t, nominal money balances D_t and labor N_t. Preferences are time separable with discount factor β and felicity

$$\frac{1}{1-\frac{1}{\sigma}}\left(C_t^{1-\frac{1}{\eta}}+\omega\left(D_t/P_t\right)^{1-\frac{1}{\eta}}\right)^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}}-\frac{\psi}{1+\varphi}N_t^{1+\varphi}, \tag{1}$$

where P_t is the price level, that is, the price of consumption goods in terms of money. Moreover, η is the intratemporal elasticity of substitution between consumption and real balances and σ is the intertemporal elasticity of substitution between bundles at different dates. If $\sigma = \eta$, utility is separable in consumption and real balances.

The New Keynesian model is usually derived by assuming separable utility. Most of our theoretical results – in particular on determinacy and the dampening of policy effects – already obtain in this case. We nevertheless develop the model for general nonseparable utility. We then emphasize below the case $\sigma > \eta$, where consumption and real balances are complements (that is, the cross partial derivative of the utility function is strictly positive). Complementarity helps fit the response of velocity to interest rates in the data. Moreover, it introduces a "cost channel" – marginal cost increases with the opportunity cost of money – which has interesting theoretical effects, as discussed in Section 2.3 below.[4]

Money is provided by the central bank which issues a digital currency that pays the nominal interest rate i_t^D. The household can also invest in other short safe assets that pay the nominal interest rate i_t^S. The cost of liquidity $i_t^S - i_t^D$ is the convenience yield on digital currency. We refrain from calling i_t^S the interest rate on short bonds. The banking models below introduce short bonds explicitly; in equilibrium, they are held by banks whose valuation pushes the bond rate below i_t^S. Instead we refer to i_t^S as the *shadow rate*. It represents the (nominal) short rate in the household's stochastic discount factor and hence the first-order term in the nominal rate of return on any asset *held directly by households*. Since we linearize the model below and abstract from higher order terms, i_t^S is the relevant rate of return for all intertemporal decisions, as well as for the valuation of firms by shareholders.

[4]We focus on utility that is homogeneous of degree one in consumption and money in order to obtain a unitary income elasticity of money demand. Some derivations of the standard model instead work with separable utility that allows for different curvature parameters. It will become clear below how to extend our results to this case.

The household budget constraint at date t is

$$P_tC_t + D_t + S_t = W_tN_t + T_t + \Pi_t + D_{t-1}(1 + i_{t-1}^D) + S_{t-1}(1 + i_{t-1}^S). \tag{2}$$

Income on the right-hand side consists of labor income at the nominal wage W_t, government transfers T_t, profits Π_t from firms, as well as payoffs from money and other assets that earn the rate i_{t-1}^S. Spending on the left hand side consists of consumption expenditure as well as a new portfolio of money and other assets. Our timing convention is that money chosen at date t provides liquidity services at that date – that is, it facilitates shopping for consumption C_t.

Firms. The supply side of the model is standard. Competitive firms make the consumption good from a continuum of intermediate goods; their production function is CES with elasticity of substitution ϵ. Monopolistically competitive firms make intermediate goods from labor using the linear production function $Y_t = N_t$. We assume Calvo price setting: the opportunity for an intermediate goods firm to reset its nominal price is an i.i.d. event that occurs with probability $1 - \zeta$. The firm commits to satisfy demand at its posted price every period.

Government and Equilibrium. The government has two policy tools: the interest rate on money i_t^D and the money supply D_t, the total size of the household's digital currency account. We specify feedback rules for these instruments below. Throughout, we consolidate the central bank and Treasury, and assume that the government levies lump sum taxes T_t to satisfy its budget constraint $D_t + P_tT_t = (1 + i_{t-1}^D)D_{t-1}$. An equilibrium then consists of sequences for consumption, labor, lump sum taxes, output of the various goods as well as the nominal interest rates i_t^S, i_t^D, the wage and the price level such that households and firms optimize, the government budget constraint and policy rules are satisfied, and the markets for goods, labor and money clear.

We consider Taylor rules for the interest rate on money, that is, i_t^D is a function of current inflation and output. The central bank thus targets the rate on an asset that earns a convenience yield, as in the banking models of Section 3. We show below that a Taylor rule on short bonds held by banks – as currently used by many central banks – works similarly to a Taylor rule for i_t^D. For the money supply, we consider rules of the form

$$\frac{D_t}{P_t} = D_t^r + \mu \left(\frac{D_{t-1}}{P_t} - D_t^r \right). \tag{3}$$

where $D_t^r > 0$ and $\mu < 1$. If $\mu = 0$, the government simply commits to a path for *real* balances. Positive μ captures short term nominal rigidity in the money supply: while inflation can temporarily erode the supply of real balances, the government gradually steers that supply towards its desired path D_t^r. If D_t^r is constant, then it represents real balances in a steady state with zero inflation.

This class of money supply rules is motivated by our banking model below. The case $\mu = 0$ is relevant when inside money is largely backed by real assets of the banking system that do not respond to inflation. To motivate the case $\mu > 0$, suppose that money is backed by long term nominal debt, part of which matures and is replaced by new issues every period. The real value of this debt – and hence the money it can back - then depends not only on the real amount of new issues, but also on how much inflation revalues the legacy long term nominal debt. We view the rule (3) as a simple reduced form way to capture this idea: a higher parameter μ corresponds to longer maturity of the debt and hence the nominal rigidity in the money supply.[5]

Equilibrium. Regardless of the details of policy, characterization of equilibrium is routine and relegated to Appendix A.1. The equilibrium paths of output, the shadow rate i_t^S, and the price level satisfy a system of difference equations: a New Keynesian Phillips curve – derived from firms' optimal price setting – together with an Euler equation (A.5) and market clearing for money. A convenient way to describe equilibrium dynamics is to linearize the difference equations around a steady state – this is how we proceed below.

Steady state. An equilibrium with constant real quantities and rates of return obtains if the government chooses constant policy parameters μ, D^r and i^D. Let π denote the steady state rate of inflation. The standard intertemporal Euler equation then implies a steady state shadow rate $i^S = \delta + \pi$, where $\delta = 1/\beta - 1$ is the household's discount rate. Output and inflation are determined by two equations. First, regardless of policy, output depends in familiar fashion on firms' markup and marginal cost:

$$Y = \left(\frac{\epsilon - 1}{\epsilon}\frac{1}{\psi}Q^{-(1-\frac{\eta}{\sigma})}\right)^{\frac{1}{\varphi + \frac{1}{\sigma}}}; \quad Q = \left(1 + \omega^{\eta}\left\{\frac{\delta + \pi - i^D}{1 + \delta + \pi}\right\}^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{4}$$

Here Q is the steady state price index for a bundle of consumption and liquidity services provided by money. It is a weighted average of the price of consumption and households' cost of liquidity, or opportunity of money – the term in braces. In the separable case, $\eta = \sigma$, firms' marginal cost does not depend on the cost of liquidity. In contrast, when $\eta < \sigma$ so money and consumption are complements, cheaper liquidity lowers firms' marginal cost and hence

[5] We emphasize that none of our money supply rules provides a "nominal anchor" for the economy, in contrast to, say, monetarist models, or a New Keynesian model with a nominal money growth rule. In fact, we have explicitly excluded the case $\mu = 1$ to ensure that the steady state quantity of money is only fixed in real terms.

increases output.[6]

Second, households must be willing to hold all real balances the government supplies. How prices adjust to make this happen depends on the money supply rule. Suppose first the government commits to a path for real balances. Market clearing for money means

$$\omega^{\eta}\left(\frac{\delta+\pi-i^{D}}{1+\delta+\pi}\right)^{-\eta} Y = d^{*}\frac{1-\mu}{1-\frac{\mu}{1+\pi}}. \tag{5}$$

Here "money demand" on the left-hand side follows from households' first order conditions for money and consumption. It is proportional to output since utility is homothetic. It is also decreasing in the opportunity cost of money; the elasticity of substitution η between consumption and real balances serves as the elasticity of money demand with respect to households' cost of liquidity.

Interest rate targeting works very much like in the standard New Keynesian model: for any target inflation rate $\pi > \mu - 1$, there is a unique interest rate $i^D < \delta + \pi$ such that π is steady state inflation; in addition, many price levels are consistent with steady state.[7] A key difference to the standard model is that the natural rate of interest lies below the rate of time preference. In order to achieve zero inflation, the government must set the interest rate to a low rate that takes into account the convenience yield conveyed by money.

Linearized model To study the dynamics of the model, we follow the standard approach of log-linearizing around a steady state with zero inflation. The inflation rate is $\Delta p_t = \log P_t/P_{t-1} = p_t - p_{t-1}$. We indicate log deviations from steady state by hats. We arrive at a system of linear difference equations for output, the interest rate and the price level. Derivations are provided in Appendix A.1. In particular, the New Keynesian Phillips Curve and Euler equation take the standard form

$$\Delta\hat{p}_t = \beta E_t\Delta\hat{p}_{t+1} + \lambda\left(\left(\varphi+\frac{1}{\sigma}\right)\hat{y}_t + \left(1-\frac{\eta}{\sigma}\right)\frac{\chi}{\delta-r^D}\left(i_t^S-\delta-(i_t^D-r^D)\right)\right), \tag{6a}$$

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma\left(i_t^S - E_t\Delta\hat{p}_{t+1}-\delta\right) + (\sigma-\eta)\frac{\chi}{\delta-r^D}E_t\left(\Delta i_{t+1}^S - \Delta i_{t+1}^D\right). \tag{6b}$$

In the separable case, $\eta = \sigma$, the last term in both equations is zero and we obtain the standard three equation model. As usual, the parameter $\lambda = (1-\zeta)(1-\beta\zeta)/\zeta$ measures the response

[6] If it were costless to produce real balances, then it would be optimal to drive the cost of liquidity to zero. In this paper, we are interested in the response of the economy with standard preferences and interest rate policies. We thus maintain preferences that preclude the possibility of satiation with money.

[7] Substituting for Y in (5) from (4) and multiplying by the bracket on the right and side makes the left hand side an upward sloping function of i^D on the left hand side that converges to infinity as i^D goes to $i^D + \pi$ and to zero as i^D goes to $-\infty$.

of inflation to marginal cost.

With complementarity, $\eta < \sigma$, there is a cost channel: a temporarily higher cost of liquidity $i_t^S - i_t^D$ increases firm's marginal cost and lowers output. The strength of the cost channel depends on the parameter χ, the elasticity of the price of a bundle of consumption and money (A.2) with respect to the cost of liquidity:

$$\chi = \left(1 + \omega^{-\eta} \left(\frac{\delta - r^D}{1 + r^D}\right)^{\eta - 1}\right)^{-1}. \tag{7}$$

The elasticity χ is positive and increasing in households' preference for liquidity as captured by the utility weight ω. In the relevant case of a strong income effect ($\eta < 1$), it is also increasing in the steady state price of liquidity chosen by the central bank: a higher price of liquidity increases the expenditure share on liquidity.

Equilibrium in the money market is summarized by the intratemporal Euler equation (A.1)

$$i_t^S - \delta = i_t^D - r^D + \frac{\delta - r^D}{\eta}\left(\hat{p}_t + \hat{y}_t - \hat{d}_t\right). \tag{8}$$

The general principle here is that, to first order, expected returns on all assets are equated. The return on money has a pecuniary component, the interest rate i_t^D on money, as well as a convenience yield. The coefficient in front of velocity is the inverse semielasticity of money demand with respect to the cost of liquidity. It depends both on the elasticity η and on the steady state spread $\delta - r^D$. If money demand is less elastic, then fluctuations in velocity have a stronger effect on the return on money.

We consider policy rules for the interest rate and real balances

$$i_t^D = r^D + \phi_\pi \Delta\hat{p}_t + \phi_y \hat{y}_t + u_t, \tag{9a}$$

$$\hat{d}_t - \hat{p}_t = \mu\left(\hat{d}_{t-1} - \hat{p}_{t-1} - \Delta\hat{p}_t\right), \tag{9b}$$

where u_t is an interest rate policy shock. We do not claim that these policy rules are optimal or otherwise desirable for CBDC rates or quantities. In particular, we are interested in the Taylor rule (9a) only because it is a simple rule that has been widely studied. Our goal is to describe what happens if the central bank targets an asset with a convenience yield in this way. The bank models below will show that this is a useful way to think of postwar monetary policy.

To define recursive equilibrium, it is helpful to work with real balances as a state variable rather than, say, the price level. An equilibrium thus consists of sequences for inflation, output, the shadow rate i_t^S, the interest rate on money i_t^D, and real balances $\hat{d}_t - \hat{p}_t$ that satisfy (6), (8) and (9). Given such sequences and some initial steady state level of prices (together with an associated initial *nominal* money supply), we also obtain paths for the money supply and the price level.

2.2 The separable case

In this section, we study the CBDC model when utility is separable in consumption and money. The behavior of the private sector is then exactly the same as in the standard three equation New Keynesian model: the New Keynesian Phillips curve and Euler equation are given by (6) with $\eta = \sigma$. Moreover, money market equilibrium (8) is the same here as in the derivation of the standard model in Woodford (2003) and Gali (2008).

The only difference between the CBDC model and the standard New Keynesian model is in the specification of policy. The standard model adds an interest rate rule for the shadow rate i_t^S and sets the interest rate i_t^D to zero. Since the central bank targets two interest rates, there cannot be an exogenous path or rule for the money supply. Instead, money is elastically supplied to achieve the desired interest rates i_t^S and i_t^D. The CBDC model, in contrast, does not impose a policy rule for the shadow rate, and it replaces the peg of i_t^D at zero with a policy rule for i_t^D. Since it drops one equation, it has to add one as well, namely the feedback rule for real balances. The money supply thus becomes a policy instrument together with the interest rate i_t^D. We now consider the implications of this change for price level determinacy as well as the transmission of policy.

Price level determinacy. When interest rate policy is specified as a path for the shadow rate i^S, the standard model is known to permit multiple equilibria, even when attention is restricted only to bounded paths for output and inflation. It is helpful to recall the intuition for this result. We focus on the case where the central bank pegs the nominal shadow rate i^S to some fixed number. One equilibrium is always that inflation and output are constant at their steady state values, so the price level remains at its initial condition. However, there are other equilibria with self-fulfilling booms and inflation.

To construct such an alternative equilibrium, suppose agents believe that output is high today and gradually falls back towards the steady state. According to the New Keynesian Phillips curve, paths of high output imply paths of marginal cost above steady state, and hence inflation. However, with a nominal interest rate peg for i^S, a path with high inflation is a path of low real expected returns on savings. According to the Euler equation, agents respond to low expected returns by intertemporally substituting consumption toward the present. High demand for goods in turn calls for high equilibrium output: the initial belief in high output is thus self-fulfilling.

A Taylor rule with a high coefficient on inflation breaks the argument: in response to high inflation, the central bank aggressively raises the nominal shadow rate and hence the real return on savings. It thereby discourages consumption today – this is what rules out a self-fulfilling inflationary boom. The central bank can achieve a similar stabilizing effect if it increases the nominal rate in response to high output. Both features of policy implement the *Taylor principle*: the response of the nominal return on savings to inflation should be larger than one.

In the CBDC model, the Taylor principle can be satisfied even if the central bank pegs the policy rate. This is because the nominal return on savings is not controlled by the central bank but moves endogenously with the convenience yield. The movement is stabilizing: for example, an inflationary boom implies higher spending and hence a higher convenience yield. It thus also raises the shadow rate as returns are equated in equilibrium according to (8). It remains to assess when the convenience yield effect is strong enough to rule out multiple equilibria. The key issue is whether an increase in spending generated by an inflation boom sufficiently increases velocity.

We say that equilibrium is locally determinate if the difference equation describing it has a unique bounded solution for any initial condition. We characterize local determinacy with feedback rules for the policy rate and the money supply by

Proposition 2.1: Suppose utility is separable in consumption and money ($\sigma = \eta$). The system of difference equations consisting of (6), (8) and(9) has a unique bounded solution for any initial level of real balances ($\hat{d}_{-1} - \hat{p}_{-1}$) if and only if

$$LR(i^S, \Delta p) := \frac{\delta - r^D}{\eta} \left(\frac{\mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} \right) + \phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} > 1. \tag{10}$$

The proof is in Appendix A.2. It is not essential for the argument that policy actively responds to inflation or output: the condition also covers the case of an interest rate peg $\phi_\pi = \phi_y = 0$. In fact, it extends to any bounded exogenous paths of the interest rate or real balances because it relies only on the eigenvalues of the homogenous part of the difference equation. The condition formally generalizes the Taylor principle to the case of an endogenous convenience yield: it ensures that the long run response $LR(i^S, \Delta \hat{p})$ of the shadow rate to inflation is larger than one. Without a convenience yield, the first term is zero and the condition reduces to that from Bullard and Mitra (2002): a sufficiently strong reaction of the central bank to either inflation or output is necessary and sufficient to stabilize the economy. The term multiplying ϕ_y is the long run response of output to inflation: according to the Phillips curve, higher inflation must be driven by higher cost and hence requires higher output.

The new element here is that the return on savings (8) reflects the convenience yield. Higher inflation goes along with a higher convenience yield for two reasons. First, higher output, or higher real spending, increases velocity as captured by the second term in the first bracket. With some rigidity in the money supply, $\mu > 0$, high inflation further increases the convenience yield by decreasing real balances. As money becomes more scarce in real terms, its convenience yield rises. From (9b), the long run response of real balances to inflation is $-\mu/(1-\mu)$. We emphasize that stabilization here is due to short term nominal rigidities – since $\mu < 1$, there is no nominal anchor due to government commitment to some long run nominal debt path. The proposition is therefore not implied by well known results on determinacy of equilibrium with a nominal anchor.

In contrast, the convenience yield effect reduces the scope for multiple equilibria even in the extreme opposite case of fixed real balances, $\mu = 0$. It now works only through changes in output – its strength thus depends on the slope of the Phillips curve. In particular, there is less scope for multiple equilibria if prices are less flexible or preferences over labor and consumption are such that marginal cost responds less to output. In either case, lower inflation implies a larger long run drop in output and hence in the convenience yield and the return on savings. From (8), the strength of the convenience yield effect also increases with the inverse semielasticity of money demand $(\delta - r^D)/\eta$, which determines by how much lower output lowers the convenience yield.

More generally, the proposition clarifies that nominal rigidities in money supply are a stabilizing force. For $\mu \in (0,1)$, the government does not commit to a path for money going forward. However, there is always a legacy amount of nominal money in the economy. If the price level falls, then this legacy money is revalued and the convenience yield declines. As in part (a), we then have a stronger stabilizing force as the convenience yield responds to the price level. We view this case as especially relevant since it suggests that simply the use of nominal money as a medium of exchange induces a stabilizing force. In other words, what matters is only that the money supply is partly predetermined from the past; it is not essential that it will not respond to future inflation.

Monetary policy transmission. We emphasize two differences between policy transmission in the CBDC model versus the standard New Keynesian model. Consider first the role of money. In the standard model, there is a strong sense in which money doesn't matter: for a given interest rate rule, money demand shocks have no effect on inflation, output and the shadow rate. Formally, the result follows because a system consisting of (6) with $\eta = \sigma$, (8) and a Taylor rule for i^S is block recursive: we can solve for output, inflation and the shadow rate independently of the parameters and any shifters of the money market equilibrium condition. The latter only determines how much money needs to be endogenously supplied in order to achieve the target interest rate i^S.

In the CBDC model with a policy rule for the interest rate on money, money matters even if utility is separable. Indeed, the system consisting of (6) with $\eta = \sigma$, (8) and (9a) is not block recursive. A shock to money demand, such as a change in the weight on money in utility, would enter as an additive shock in (8). If the central bank sticks to its interest rate rule, such a shock affects the shadow rate i_t^S and hence the allocation. At the same time, a change in the exogenous quantity of digital currency supplied by the central bank has real effects for a given interest rate rule. In the banking models studied below, this property carries over to the quantity and quality of collateral assets used by banks to back inside money.

Second, consider interest rate policy. In the CBDC model, changes in the policy rate have weaker real effects than in the standard New Keynesian model. The reason is the imperfect pass-through from the policy rate to the shadow rate, and hence to intertemporal decisions, as described by (8). Indeed, consider a positive monetary policy shock, say, that increases the nominal rate on money. With sluggish price adjustment, the real rate on money also increases, which entails lower output and lower inflation on impact, as in the standard model. However, lower spending also reduces the convenience yield on money. As returns on all assets are equated according to (8), the effect of the policy shock on the shadow rate i^S is lower than in the standard model. In this sense, interest rate policy is weaker. We quantify the effect in Section 2.4.

2.3 Nonseparable utility and the cost channel

In the CBDC model, the pass-through (8) from the policy rate to the shadow rate depends importantly on the elasticity of money demand η. Since standard estimates of η are lower than conventional numbers for the intertemporal elasticity of substitution σ, the separable case is overly restrictive. In this section, we thus explore the nonseparable case with $\eta < \sigma$, where money and consumption are complements in utility. A key new feature is then that the cost channel terms in the Phillips curve and Euler equation become relevant: a temporarily higher cost of liquidity for households $i_t^S - i_t^D$ increases firms' marginal cost and hence inflation; at the same time, it makes consumption more expensive and hence lowers output.

The introduction of a cost channel accentuates the difference between interest rate policy in the CBDC model versus the standard model. To see this, consider again an increase in the policy rate in the CBDC model. A drop in spending and hence a lower convenience yield now feeds back to output and inflation: a lower cost of liquidity amplifies the fall in inflation but further dampens the fall in output. Interestingly, the cost channel effects here are the opposite of those in the standard model where the central bank increases the shadow rate holding fixed the rate on money, so the cost of liquidity for households increases. In the standard model, the cost channel thus dampens the fall in inflation and amplifies the fall in output.

The presence of a cost channel in the CBDC model also introduces a new source of fragility if the central bank responds too strongly to output. Indeed, suppose that households believe in a path of high expenditure on *bundles* of consumption goods and liquidity. Along such a path, cost is high for firms which translates into high inflation. With a low enough return on savings, the path is self-fulfilling. The new feature is that such a path need not exhibit high output. Instead, spending by household and firms' cost could be high because liquidity is expensive, while output is actually below steady state. We thus have self-fulfilling stagflation.

With a strong cost channel, an interest rate policy that responds positively to output can be destabilizing. To rule out multiplicity, we would like to follow the Taylor principle and increase the nominal return on savings when inflation is high. However, with a threat of stagflation, it does not help to lower the policy rate when output falls. If interest rate policy responds too strongly to output, then the above dynamics can be explosive and no bounded equilibrium exists. To rule out this case, we assume in what follows that

$$\phi_y \left(\frac{1}{\eta} - \frac{1}{\sigma} \right) \frac{\chi}{\varphi + \frac{1}{\sigma}} < \frac{\eta}{\delta - r^D}. \tag{11}$$

The condition is always satisfied if $\phi_y = 0$ or there is no cost channel. More generally, it restricts the product of ϕ_y and the long run effect of the policy rate on output.

The key to local determinacy is again the long run response of the return on savings to inflation. With a cost channel, it becomes

$$\begin{aligned} LR(i^S, \Delta \hat{p}) = & \frac{\delta - r^D}{\eta} \left(\frac{\mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} \right) + \phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} \\ & + \left(\frac{1}{\eta} - \frac{1}{\sigma} \right) \frac{\chi}{\varphi + \frac{1}{\sigma}} \left(\phi_\pi - 1 - \phi_y \frac{\mu}{1-\mu} \right). \end{aligned}$$

The first line takes the same form as in the separable case (10), although now we have $\eta < \sigma$. The second line shows that, with a cost channel, an aggressive interest rate response to inflation still helps avoid multiplicity, whereas an aggressive response to output now hurts. A peg leaves more room for multiplicity since the effect of inflation on output is weaker and hence the convenience yield effect is reduced. Finally, (11) implies that more nominal rigidity (higher μ) contributes to stability as before.

The determinacy properties with feedback rules are summarized by:

Proposition 2.2: Suppose consumption and money are complements in utility ($\eta < \sigma$) and (11) *holds.*

The system of difference equations consisting of (6), (8) and (9) has a unique bounded solution for any initial level of real balances ($\hat{d}_{-1} - \hat{p}_{-1}$) if and only if $LR(i^S, \Delta p) > 1$.

The proof is in Appendix A.2.

2.4 Numerical example

In this section, we present a numerical example to show that the differences between the standard model and the CBDC model can be quantitatively large.

Calibration. The model period is a quarter. We select a discount factor of $\beta = 0.99$, which implies a 4 percent discount rate δ per year. To calibrate the discount rate and the opportunity cost of money, we need measures of the interest rate on money as well as the shadow rate in the households' stochastic discount factor. For the former, we choose the interest rate on Money of Zero Maturity (MZM), a broad measure of money constructed by the St. Louis Fed. For the latter, we want a short rate that is not contaminated by the convenience yield effects we study in our bank models below. We thus use the 3 month rate of the yield curve constructed by Gurkaynak, Sack and Wright (2007) using only Treasury bonds, leaving out T-bills that are predominantly held by payment intermediaries. The resulting average deposit spread is 2.4% per year, so we work with an average deposit rate $r^D = .004$.

We follow standard practice to identify the elasticity of money demand η from the time series relationship between the velocity of money and its opportunity cost. In particular, we find the semielasticity $\eta / \left(\delta - r^D\right)$ by regressing log velocity of MZM on the spread between the 3 month T-bill rate and the MZM own rate, which is the average rate on instruments in MZM. The coefficient on the spread is 8.1 which implies an elasticity of $\eta = .22$. This number is similar to what has been used in past studies. We identify the final preference parameter $\omega = 0.14$, the weight on money in utility from (5), to match an average velocity of 1/2.

Other parameters take standard values from the New Keynesian literature. We set both the intertemporal elasticity of substitution σ and the Frisch elasticity φ equal to one. The probability of resetting prices is $1 - \zeta = .75$, so the response of inflation to marginal cost is $\lambda = .085$. Without a cost channel, this response only consists of the response of inflation to output, given by $\lambda \left(\varphi + 1/\sigma\right) = .17$. The strength of the cost channel is then measured by the parameter $\chi = .0118$; in other words, a one percentage point increase in the cost of liquidity has about the same effect on inflation as a 70bp increase in output.

For policy, we assume that the central bank keeps real balances constant, that is, we set $\mu = 0$ in the money supply rule (9b). For interest rate policy, we assume a Taylor rule with interest rate smoothing that is consistent with typical estimates. In our CBDC Taylor rule, the current policy rate depends not only on inflation but also on the last policy rate according to

$$i_t^D = .5 i_{t-1}^D + 1.5\pi_t + v_t. \tag{12}$$

Moreover, we use a version of the standard model where money pays a constant interest rate r^D. This nonstandard assumption has no effect on dynamics. It permits a cleaner model comparison in the sense that the average interest rate on money and the average cost of liquidity for households are the same across the two models.

Dampening. Figure 1 considers responses to an unanticipated increase in the policy rate by 25bps, or 1 percentage point per year. The top three panels report percentage deviations from steady state in the price level, output and nominal money. The bottom three panels report percentage *point* deviations from steady state in inflation, the policy rate and households' cost of liquidity, that is, the spread between the shadow rate and the deposit rate. In all panels, light gray and black lines represent the standard New Keynesian model and the CBDC models, respectively.

The impact effects illustrate the dampening of interest rate policy when the policy instrument earns a convenience yield. While contractionary policy causes a recession and deflation in both models, output and inflation responses in the CBDC model are only about half the size of those in the standard model. There are two reasons, illustrated in the bottom right panel. First, pass-through is imperfect in the CBDC model: the spread between the policy rate and the shadow rate declines. This effect is quantitatively relatively small. Second, the cost of liquidity in the standard model moves in the opposite direction from the CBDC model. This is an important force that makes output fall much more in the standard model.

3 Banks, the central bank and fixed income markets

In this section we describe a simple model of the banking sector and the central bank. Banks issue debt and equity, and hold assets, including reserves. What distinguishes banks from other firms is that their debt provides liquidity services. This means that (i) bank debt—labelled *deposits* in what follows—enters households' utility as money and (ii) banks must handle liquidity shocks that are proportional to their level of deposits. Our model further assumes that banks can issue equity at no cost every period and that all debt is short term, so banks effectively behave myopically. For simplicity, we work directly with two-period-lived banks and describe partial equilibrium in fixed income markets *within a period*. In the next section we embed this "banking module" into the New Keynesian model.

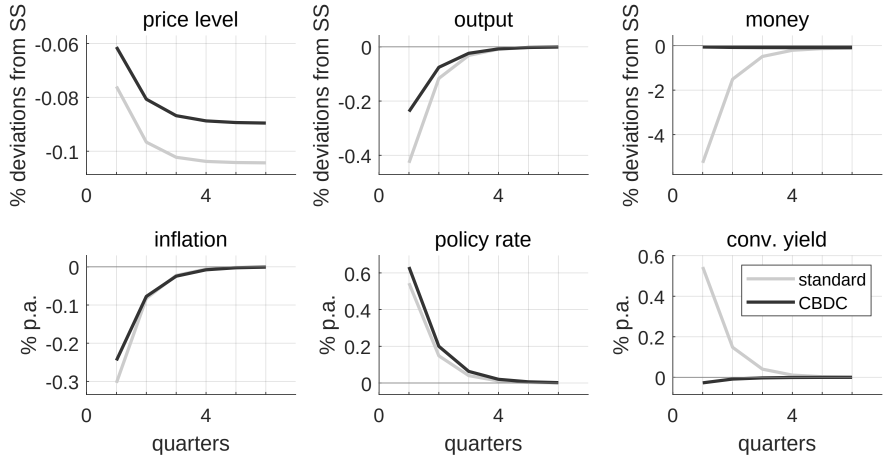

Figure 1: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate.

3.1 Setup

Banks hold reserves M_t, outside money issued by the government that earns the nominal interest rate i_t^M. They issue deposits D_t, inside money held by households that earns the interest rate i_t^D. They may also take positive or negative position in an overnight interbank market, or "Federal funds" market. The balance sheet of the typical bank is

Assets		Liabilities	
M	Reserves	Money	D
F^+	interbank lending	interbank borrowing	F^-
A	Other assets	Equity	

Other assets A_t available to banks earn the nominal interest rate i_t^A. Banks maximize shareholder value. We assume that bank equity can be adjusted every period at no cost.

Liquidity shocks. To generate a liquidity benefit for reserves, we introduce bank level liquidity shocks, motivated by banks' provision of liquid inside money. Formally, suppose every period has two subperiods. In the first subperiod, bank i selects a portfolio of reserves M_t^i and other assets A_t^i and issues money D_t^i. In the second subperiod an individual bank must transfer $\tilde{\lambda}_t^i D_t^i$ funds to other banks. If $\tilde{\lambda}_t^i$ is negative, then the bank receives funds and thus increases its debt. We assume that liquidity shocks are iid across banks with a continuous cdf $G\left(\tilde{\lambda}_t^i\right)$ that is strictly increasing on the interval $[-\bar{\lambda}, \bar{\lambda}]$., with $G(-\bar{\lambda}) = 0$ and $G(\bar{\lambda}) = 1$. We also assume that liquidity shocks have mean zero. With a continuum of identical banks, this means that all flows in the second subperiod remain within the banking system.

Once liquidity shocks have been realized, the interbank loan market opens. Interbank loans are traded competitively at the rate i_t^F; they are repaid in the first subperiod of the subsequent period. Markets for deposits, other assets or equity remain closed. The bank budget constraint in the second subperiod is therefore

$$M_t^i - \tilde{\lambda}_t^i D_t^i = \tilde{M}_t^i + F_t^{i+} - F_t^{i-}, \tag{13}$$

where $\tilde{M}_t^i$ denotes reserves held overnight (carried over to period $t+1$), while F_t^{i+} and F_t^{i-} are funds lent and borrowed in the fed funds market, respectively.

Leverage constraint. Banks can issue debt only if they have sufficient collateral to back it, as described by the leverage constraint

$$F_t^{i-} + D_t^i \left(1 - \tilde{\lambda}_t^i\right) \leq \ell \left(\tilde{M}_t^i + \rho_F F_t^{i+} + \rho_A A_t^i\right). \tag{14}$$

where ℓ, ρ_F and ρ_A are positive scalars. The parameter $\rho_A < 1$ captures the idea that other assets are worse collateral than reserves. Similarly, we assume that $\rho_F < 1$: interbank loans—claims on the private sector—are also worse collateral than reserves. This assumption makes it worthwhile for banks to hold reserves even if the fed funds rate i_t^F is above the reserve rate i_t^M. The parameter $\ell \leq 1$ serves as a bound on leverage, defined as the ratio of debt to quality-weighted assets. One interpretation of the constraint is as a capital requirement: equity must be higher if assets are lower quality. Even without regulation, a leverage constraint can be viewed as a simple way to model an increasing marginal cost of debt.[8]

When a bank decides on its balance sheet in the first subperiod, it has to choose enough initial reserves and other assets to withstand the largest possible inflow without taking any interbank positions. The reason is that interbank positions (non-negative F_t^{i-} and F_t^{i+}) add to debt and are worse collateral than reserves; interbank positions thus do not help relax the leverage constraint (14). Suppose a bank experiences the largest possible inflow and holds $\tilde{M}_t^i = M_t^i + \bar{\lambda} D_t$ from (13). This bank faces the *worst case leverage constraint*

$$\left(1 + \bar{\lambda}\left(1 - \ell\right)\right) D_t^i \leq \ell \left(M_t^i + \rho_A A_t^i\right). \tag{15}$$

[8]In a more general model, such costs might be derived from deadweight costs of bankruptcy. Collateral quality can then be derived from the riskiness of bank assets. While the resulting tradeoffs that determine leverage are similar to the ones here, adding portfolio choice under risk yields additional testable predictions on balance sheet ratios, explored for example in Lenel, Piazzesi and Schneider (2019).

A bank that satisfies (15) also satisfies its leverage constraint for any other net inflow $\tilde{\lambda}_t^i > -\bar{\lambda}$.

Bank cash flow. Bank i's nominal cash flow in the first subperiod reflects changes in reserves, deposits, and other asset positions as well as interest on those positions and payoffs from trading in the interbank loan market:

$$\begin{aligned}&\tilde{M}_{t-1}^i\left(1+i_{t-1}^M\right)-M_t^i-D_{t-1}^i\left(1-\tilde{\lambda}_{t-1}^i\right)\left(1+i_{t-1}^D\right)+D_t^i\\&+A_{t-1}^i\left(1+i_{t-1}^A\right)-A_t^i+\left(F_{t-1}^{i+}-F_{t-1}^{i-}\right)\left(1+i_{t-1}^F\right).\end{aligned} \tag{16}$$

An individual bank maximizes the present value of cash flow, discounted at the shadow rate i_t^S. Since the model is deterministic, the short rate i_t^S in the household stochastic discount factor serves as the banks' cost of capital, or the required rate of return on bank equity. It is convenient to work with nominal cash flows discounted by nominal rates to avoid extra notation.

Imperfect competition in deposit markets. To allow for bank market power, we assume monopolistically competitive banks that offer varieties of deposits. We thus modify preferences relative to Section 2 so households care about a CES aggregate of different varieties D_t^i, each produced by a different bank i:

$$D_t=\left(\int\left(D_t^i\right)^{1-\frac{1}{\eta_b}}\right)^{\frac{1}{1-\frac{1}{\eta_b}}},$$

where η_b measures the elasticity of substitution between varieties. One interpretation is that the household sector works like a large "family" with members in different regions, and for historical reasons banks exert local market power. The key effect we are after is that deposits are a cheap funding source for banks not only because of their liquidity benefit to households, but also because of market power.

Consider deposit demand faced by an individual bank. Bank i supplies liquidity to households at the price $(i^S-i_t^{D,i})/(1+i_t^S)$, where $i_t^{D,i}$ is the deposit rate promised by bank i. CES preferences imply an ideal price index that aggregates the individual liquidity prices. We define the average deposit rate i_t^D such that the spread $(i^S-i_t^{D,i})/(1+i_t^S)$ achieves that aggregate price of liquidity. We can then write deposit demand as

$$D_t^i=\left(\frac{i_t^S-i_t^{D,i}}{i_t^S-i_t^D}\right)^{-\eta_b}D_t. \tag{17}$$

The derivation is familiar from monopolistic competition in the goods market and relegated to Appendix A.3.1. The only unusual feature is that prices take the form of spreads since the relevant good is liquidity.

Partial equilibrium in fixed income markets. In the remainder of this section, we study the behavior of the banking system and the central bank at date t taking as given the interest rate policy of the central bank. Banks are ex ante identical and their objective function and constraints are linear in their balance sheet positions. As a result, only aggregate ratios are determinate. We thus define a (symmetric, partial) equilibrium in the banking sector as initial balance sheet ratios M_t/D_t and A_t/D_t, a deposit interest rate i_t^D, a rate on other assets i_t^A, as well as a distribution of interbank market positions F_t^+/D_t and F_t^-/D_t such that banks optimize and the date t markets for interbank loans, reserves, money and other assets clear.

3.2 Bank optimization

With a positive deposit spread $i_t^S - i_t^D > 0$, the leverage constraint has to bind in at least some states of the world. From the perspective of the bank, inside money represents a source of funding that is strictly cheaper than equity, which must earn the shadow rate i_t^S. Without a leverage constraint, it would thus be optimal to fund the bank entirely with inside money. The case of positive deposit spread is relevant since we already know from the household Euler equation (A.1) that deposits provide a convenience yield whenever the supply of real balances is finite. In equilibrium with banks, a limited quantity of collateral will imply a limited quantity of inside money, which in turn justifies a positive deposit spread.[9]

Optimal liquidity management. In the second subperiod, a bank chooses the allocation of funds to reserves or interbank positions on the right hand side of the budget constraint (13) for the second subperiod. It maximizes the contribution of those positions to shareholder value (16), subject to the budget constraint as well as the leverage constraint (14).

The presence of a leverage constraint implies that banks have to hold reserves overnight if they experience a large inflow of inside money (and hence reserves). Consider a bank with a net inflow large enough so the left hand side of the budget constraint (13) is positive. The bank can either lend out reserves or hold them overnight on its balance sheet. The difference is that interbank loans are worse collateral than reserves, since $\rho_F < 1$. Lending out all reserves may thus lower collateral sufficiently so the bank cannot back its debt—which has also increased with the inflow—and violates the leverage constraint (14). In particular, the bank must hold

[9] A hard leverage constraint simplifies the analysis, but is not essential for our results. In Piazzesi and Schneider (2018), optimal leverage follows from a smooth tradeoff between the marginal cost of leverage and the liquidity benefit of deposits. The key point both here and in that model is that the liquidity benefit works like the tax advantage of debt in the standard tradeoff theory of capital structure – combined with an increasing marginal cost of debt, it generates a determinate optimal leverage ratio.

reserves overnight if

$$\tilde{\lambda}_t^i < \lambda_t^{i*} := \frac{D_t^i - \rho_F \ell M_t^i - \rho_A \ell A_t^i}{(1-\rho_F \ell)\, D_t^i}. \tag{18}$$

The threshold shock λ_t^{i*} is the smallest shock (or the largest net inflow) so the leverage constraint is satisfied even when all reserves are lent out.[10] We also note that the leverage constraint never binds for banks that experience a large enough reserve *outflow*. To see this, consider a bank with an outflow of reserves that makes the left hand side of the budget constraint (13) negative. This bank must borrow in the interbank market and it doesn't make sense for it to hold any reserves. However, interbank borrowing counts against the leverage constraint in exactly the same way as inside money. As a result, funding an outflow of reserves via interbank borrowing does not affect debt capacity.

The optimal choice of reserves now depends on interest rates. If $i_t^F > i_t^M$, then banks would like to lend out as many reserves as possible to take advantage of the high interbank rate. They economize on reserves by holding the minimum amount that keeps the leverage constraint satisfied. If $i_t^F = i_t^M$, in contrast, banks are indifferent between holding reserves or lending them out: any reserve position above the minimum amount is also optimal. We summarize this result as

Proposition 4.1 (Demand for overnight reserve holdings). A bank's optimal reserve holdings in the second subperiod satisfy

$$\tilde{M}_t^i \geq \max\left\{\lambda_t^{i*} - \tilde{\lambda}_t^i, 0\right\} \frac{1-\rho_F \ell}{\ell\,(1-\rho_F)} D_t^i, \tag{19}$$

with equality if $i_t^F > i_t^M$.

The proof is in the appendix and follows directly from bank first order conditions. The key takeaway is that the nonpecuniary benefit of relaxing a binding leverage constraint pushes some banks to hold reserves even when they are dominated in rate of return by interbank lending. Indeed, a shareholder value maximizing bank that both lends out reserves and holds them directly equates the total returns on both positions

$$1 + i_t^F + \tilde{\gamma}_t^i \rho_F \ell = 1 + i_t^M + \tilde{\gamma}_t^i \ell, \tag{20}$$

where $\tilde{\gamma}_t$ denotes the Lagrange multiplier on the leverage constraint in the second period problem. The total return on either position consists not only of the pecuniary return – the interest rate – but also a nonpecuniary component that reflects the collateral value of the position. Because the collateral value of interbank loans is lower than that of reserves, a

[10]To derive the threshold, solve the budget constraint (13) for F_t^{+i}, substitute into the leverage constraint (14) and set $\tilde{M}_t^i = F_t^{-i} = 0$.

positive spread between the interbank and reserve rate is consistent with bank optimization.

Banks' optimal interbank market positions also depend on realized deposit inflows. Banks with large outflows must borrow in the fed funds market to satisfy their budget constraint in the second subperiod. Banks with large enough inflows do not borrow, since the increase in deposits takes up all the available debt capacity. If the interbank rate is above the reserve rate, they keep just enough reserves to still satisfy the collateral constraint. Some are able to lend out all the reserves they have, while others hold both reserves and interbank loans. If $i_t^F = i_t^M$, then the interbank market position is indeterminate.

Optimal bank portfolios and capital structure. Consider now a bank's portfolio and capital structure choice in the first subperiod. The objective function is

$$\begin{aligned}
&\left(1+i_t^A\right) A_t^i - \left(1+i_t^{D,i}\right) D_t^i + \left(1+i_t^F\right) M_t^i \\
&-\left(i_t^F - i_t^M\right) \frac{1-\rho_F \ell}{\ell\left(1-\rho_F\right)} \int_{-\bar{\lambda}}^{max\{\lambda_t^{i*}, -\bar{\lambda}\}} \left(\lambda_t^{i*} - \tilde{\lambda}\right) dG\left(\tilde{\lambda}\right)] D_t^i \\
&-\left(A_t^i + M_t^i - D_t^i\right)\left(1+i_t^S\right),
\end{aligned} \tag{21}$$

where the threshold shock λ_t^{i*} is given by (18). If either $i_t^F = i_t^M$ or $\lambda_t^{i*} < -\bar{\lambda}$, the second line disappears: banks anticipate that all liquid positions chosen in the second subperiod earn the interbank interest rate i_t^F. This may happen for two reasons. First, policy may be such that reserves simply pay the interbank rate. Alternatively, the bank might choose to acquire so much collateral in the first subperiod that it avoids holding any reserves overnight – formally, it chooses a threshold $\lambda_t^{i*} < -\bar{\lambda}$. In contrast, a bank that sets $\lambda_t^{i*} > -\bar{\lambda}$ holds reserves overnight with positive probability.

Banks choose positions M_t^i and A_t^i and the deposit rate i_t^{iD} to maximize (21) subject to the leverage constraint (15) and the demand function (17). They take as given other banks' deposit choices as well as the interbank rate that will prevail in the second subperiod – since there is no aggregate risk, they can perfectly foresee that rate. As long as $i_t^F < i_t^S$ and $i_t^A < i_t^S$, it is optimal for banks to always incur some liquidity risk, in the sense of setting $\lambda_t^{i*} > -\bar{\lambda}$. Indeed, consider a bank that sets $\lambda_t^{i*} \leq -\bar{\lambda}$. Such a bank always satisfies (15) since $\rho_F < 1$. Moreover, it loses money on its collateral holdings that earn a rate below the cost of capital i_t^S. As a result, the bank can reduce collateral positions so as to set λ_t^{i*} slightly higher than $\bar{\lambda}$. It thereby strictly lowers losses (in the first line of (21)), but incurs no marginal cost as the derivative of the second line of (21) is zero at $\lambda^{i*} = -\bar{\lambda}$.

Given its constant returns to scale business model, shareholder value maximization requires that banks equate returns on all balance sheet positions to the cost of capital i_t^S, the short rate in the household's stochastic discount factor. For $\lambda^{i*} > -\bar{\lambda}$, the FOC for reserves and other assets A are

$$i_t^S = i_t^M + \left(i_t^F - i_t^M\right)\left(1 - G\left(\lambda_t^{i*}\right)\right) + \ell\left\{\frac{i_t^F - i_t^M}{\ell\left(1-\rho_F\right)} G\left(\lambda_t^{i*}\right) + \gamma_t^i\right\}, \tag{22a}$$

$$i_t^S = i_t^A + \rho_A \ell\left\{\frac{i_t^F - i_t^M}{\ell\left(1-\rho_F\right)} G\left(\lambda_t^{i*}\right) + \gamma_t^\iota\right\}, \tag{22b}$$

respectively, where γ_t^i is the Lagrange multiplier on (15). As in the FOC for the second subperiod (20), returns on balance sheet positions consist of pecuniary components and convenience yields, the terms in braces. Reserves always earn a higher convenience yield than other assets since they are better collateral ($\rho_A < 1$).

Convenience yields on bank asset positions arise whether or not there is a positive spread between interbank and reserve rates. This is because bank assets are always valuable as collateral to back inside money. To see this, suppose that $i_t^F = i_t^M$. Liquidity management – that is, the allocation of funds to reserves or interbank positions on the right hand side of (13) – is then irrelevant for bank profits. It is therefore optimal for banks to issue as much money as the worst case leverage constraint (15) allows. In the FOC (22), all terms involving the liquidity shock distribution vanish and spreads on bank assets are proportional to the multiplier γ_t on that constraint.

If instead banks face a positive cost of liquidity $i_t^F - i_t^M > 0$, then pecuniary and nonpecuniary returns on bank assets depend on the distribution of liquidity shocks. On the one hand, the pecuniary return on reserves is stochastic: with probability $G(\lambda^*)$, banks are constrained and earn only the reserve rate instead of the higher interbank rate i_t^F. On the other hand, the shadow value of reserves as collateral – the term in braces in (22a) – depends on how often the leverage constraint binds. In particular, it sums up the expected multiplier on (14) as derived in (20), and the multiplier on (15). For other assets, the pecuniary return is simply the interest rate and the convenience yield is a share ρ_A of that on reserves, due to lower collateral quality.[11]

Deposit pricing. The bank's first order condition for deposits is

$$\frac{\eta_b - 1}{\eta_b}\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right)\frac{M}{D} + \left(i_t^S - i_t^A\right)\frac{A}{D}. \tag{23}$$

The "good" banks offer to households is liquidity, priced via the spread $i_t^S - i_t^{D,i}$. Monopolis-

[11] The same principle governs the pricing of interbank loans. Indeed, (22a) can be rearranged to take the same form as (22b), but with i_t^A and ρ_A replaced by i_t^F and ρ_F, respectively.

tically competitive banks price liquidity at a markup over marginal cost, the term in braces. Marginal cost in turn is a weighted average of spreads on the two collateral assets used to back deposits. It follows that the typical collateral asset also earns a lower pecuniary return than the short rate. Competition between banks for collateral assets implies that those assets inherit part of the liquidity benefit conveyed by deposits.

Equilibrium. In an equilibrium with a given pair of interest rates, the supply of reserves provided by the central bank must be large enough to meet banks' demand for holding reserves overnight. From Proposition 4.1, that demand comes from all banks with liquidity shocks below λ_t^*, that is, banks with large enough deposit inflows. Since liquidity shocks are iid we derive a market clearing condition by integrating over the cross section of banks to arrive at

$$\frac{1-\rho_F \ell}{\ell\left(1-\rho_F\right)} \int_{-\bar{\lambda}}^{\lambda_t^*}\left(\lambda_t^*-\tilde{\lambda}\right) dG\left(\tilde{\lambda}\right) \leq \frac{M_t}{D_t}, \tag{24}$$

with equality if $i_t^F > i_t^M$. If $i_t^F = i_t^M$, banks are banks' aggregate demand or reserves is set-valued so market clearing becomes an inequality. In either case, Walras' law implies that reserve market clearing implies that the interbank funds market also clears.

The money multiplier is negatively related to the threshold shock λ^* and hence the probability that banks have to hold cash overnight. Indeed, the derivative of the bracket on the left hand side is $G\left(\lambda^*\right) > 0$. Intuitively, if banks hold fewer reserves relative to money, then less cash is available for the sector overall to withstand liquidity shocks. As a result, the equilibrium probability of holding cash overnight must decline.

To sum up, an equilibrium in fixed income markets is characterized by the two first order conditions for bank assets (22), the deposit pricing condition (23), and the market clearing condition (24), with λ^* defined by (18). As long as (15) does not bind, these four equations determine the four unknowns $\left(A_t/D_t, M_t/D_t, i_t^A\right)$ and i^D. If (15) does bind, it serves as a fifth equation and the fifth unknown is the multiplier γ_t. We are now ready to state the main result of this section:

Proposition 4.2 (Equilibrium in fixed income markets). If the support bound of the liquidity shock distribution $\bar{\lambda}$ is sufficiently small, then

(a) for any reserve rate i_t^M and interbank rate i_t^F such that $i_t^M < i_t^F < i_t^S$, there is a unique equilibrium in fixed income markets,

(b) there is a threshold interbank rate $i_t^{F*} \in \left(i_t^M, i_t^S\right)$ such that the reserve-deposit ratio is strictly decreasing in the interbank rate i^F for $i_t^F > i_t^{F*}$ and constant at $M_t/D_t = \underline{m}$, say, for $i_t^F \leq i_t^{F*}$

(c) for any $i_t^M = i_t^F < i_t^S$, there is a continuum of equilibria in fixed income markets, indexed by an

interval for the reserve-deposit ratio $M_t/D_t \in [\underline{m}, \bar{m}]$.

(d) as the support bound $\bar{\lambda}$ *of the liquidity shock distribution converges to zero, we have* $\ell M_t/D_t + \rho^A \ell A_t/D_t \to 1$, $M_t/D_t \to 0$ *for all* $i_t^F > i_t^M$*, and, in particular,* $\underline{m} \to 0$.

The structure of equilibria is displayed in Figure 2. We show the "liquidity ratio" of reserves to deposits M/D on the horizontal axis and the interbank rate on the vertical axis. At high interbank rates, the liquidity ratio is strictly decreasing in the interbank rate. We refer to equilibria in this region as *elastic*. When we consider the full model below, they obtain when the central bank runs a corridor system: it chooses the quantity of reserves to implement a target interbank rate. At a low enough interbank rate, the liquidity ratio is still unique but no longer responds to the interest rate. Finally, when the interbank rate equals the reserve rate, many balance sheet ratios above a lower bound are consistent with equilibrium. This region is relevant in a floor system: the quantity of reserves becomes a separate policy instrument that selects equilibrium in fixed income markets.

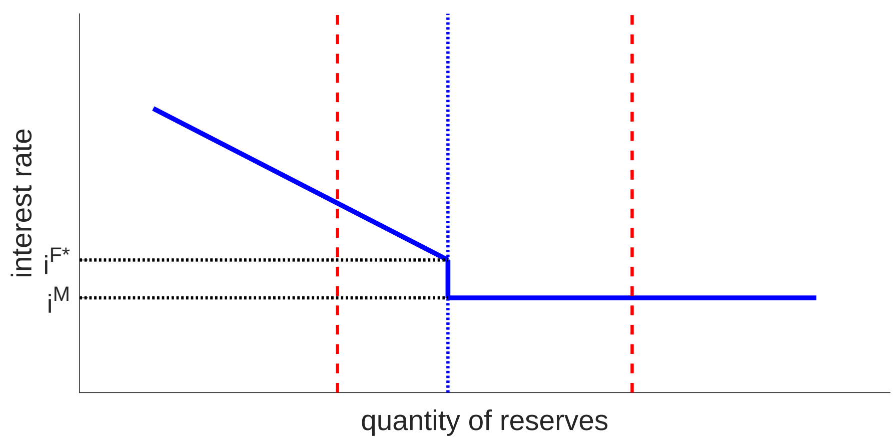

Figure 2: Structure of equilibria.

The shape of the curve in Figure 2 follows from banks' first order conditions. Formally, elastic equilibria are such that the worst case leverage constraint (15) is slack, that is, banks lend out reserves in the interbank market even when they receive the largest possible deposit inflow. Intuitively, when banks' cost of liquidity $i_t^F - i_t^M$ is sufficiently large, banks are eager to avoid it. Liquidity management is then about changing how often it binds, leading to an elastic response. At a low cost of liquidity, banks' asset positions are determined by the need to avoid the worst case leverage constraint (15). That constraint must also be respected when $i_t^F = i_t^M$. However, at a zero cost of liquidity, it is no longer crucial to hold as few reserves as possible, so any liquidity ratio that allows the bank to satisfy (15) while maintaining nonnegative holdings of other assets is also optimal.

While the shape of the equilibrium schedule depends on the relative frequency of liquidity shocks, it does not depend on their scale. It is intuitive that there must be some upper bound on the size of the shocks to make banking profitable. The important feature for our purposes is that there does not have to be a lower bound: our results hold even if liquidity shocks are very small. In a corridor system, excess reserves can therefore be tiny compared to the quantity of inside money, as in the data. In fact, our analysis below considers the "reserveless limit" where the support of liquidity shocks goes to zero.

Elastic equilibria To prepare integration with the full model, consider in more detail how banks respond to interest rates in an elastic equilibrium. We consider responses near a steady state where all rates and balance sheet ratios are constant. We can linearize the first order condition for reserves (22a) around a steady state with $\gamma_t = 0$, substitute for λ^* from its definition and for the liquidity ratio from market clearing (24). We then obtain in the neighbourhood of an elastic equilibrium

$$\hat{d}_t - \hat{a}_t^r = \varepsilon \left(\frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M} \right); \qquad \varepsilon := \frac{(1-\rho_F \ell)\, D}{\rho_A A} \frac{(1-\rho_F + \rho_F G(\lambda^*))}{1-\rho_F} \frac{G(\lambda^*)}{g(\lambda^*)}, \tag{25}$$

where all variable values are at the steady state. In particular, the statistic ε describes the interest elasticity of deposit supply. A higher policy rate i_t^F – other things equal – lowers the banks' supply of money. The strength of this effect depends on the distribution of liquidity shocks, as well as on central bank operating procedures.

Banks' response can be decomposed into two effects, captured by the two spreads in (25). First, a higher spread between the short rate and the interbank rate reflects the multipliers on the leverage constraint, or the cost of collateral. Indeed, in an elastic equilibrium, (22) implies

$$i_t^S - i_t^F = \frac{\rho_F}{\rho_A} \left(i_t^S - i_t^A \right) \tag{26}$$

In other words, interbank loans are valued as collateral with quality ρ_F. When collateral assets earn a lower return, banks optimally increase leverage in order to maintain the same return on equity.

Second, a higher spread between the interbank and reserve rates means that the cost of liquidity – the cost of holding reserves at the constraint – is higher. In response, banks optimally reduce leverage. This effect shows that the nature of the corridor implemented by the central bank matters for policy transmission. In a system where the reserve rate is pegged at, say, zero and thus far below the target interbank rate, a hike in the target that leaves the reserve rate unchanged leads to a stronger response on the part of banks. If instead the reserve rate moves in lockstep with the target – so the width of the corridor remains constant – we have a weaker effect.

4 Monetary policy with a banking system

We now turn to equilibria of the full model. Here we integrate the bank module of Section 3 with the standard household and firm behavior from Section 2. The only extra ingredients we need to specify are government policy and the supply of other assets available to banks. The government chooses paths or rules for the policy rates i_t^F and i^M as well as the supply of reserves M_t. To pin down the ratio of reserves to deposits even when $i_t^F = i_t^M$, we allow for the same class of rules for reserves that we considered for the entire money stock in (3):

$$\frac{M_t}{P_t} = M^r + \mu \frac{P_{t-1}}{P_t}\left(\frac{M_{t-1}}{P_{t-1}} - m^*\right). \tag{27}$$

In particular, the government commits to a long run quantity of real balances $M^r > 0$. We further assume that the *real* supply of other assets is given by an exogenous path A_t^r, so in equilibrium $A_t = P_t A_t^r$. Concretely, we can think of firms or the government issuing a fixed amount of debt in real terms.[12]

An equilibrium of the full model now consists of household, firm and bank plans as well as wages, prices and interest rates such that agents optimize and all goods and asset markets clear. In particular, the bank balance sheet ratios A_t/D_t and M_t/D_t as well as interest rates on other assets and deposits i_t^A and i^D are part of an equilibrium in fixed income markets, as defined in Section 3. Throughout this section, we work directly with the limiting case where the support of the liquidity shock distribution $\bar{\lambda}$ goes to zero. We know from Proposition 4.2 that real balances are then given by $D_t/P_t = M_t/P_t + \rho^A \ell A_t^r$, and that M_t/P_t is negligible if $i_t^F > I_t^M$.

4.1 Steady state with floor and corridor system

Inflation is determined by interest rate targeting: the government can implement a given inflation rate with either a corridor or a floor system. Suppose the government chooses steady state

[12] The only other element of the model that is affected is profits in the household budget constraint, which add up firm and bank profits. Since households and firms operate in frictionless equity markets, their marginal conditions are unaffected. A richer model could make the demand for bank loans endogenous, and possibly responsive to the state of the economy. We choose to work with exogenous rules to maximize transparency. Fixed debt is a baseline scenario motivated by the fact that bank assets tend to adjust slowly to shocks.

policy rates i^F and i^M. We distinguish two types of steady state equilibria based on the spread between the two. An *equilibrium with a corridor system* satisfies $i^F > i^{F*}$, so partial equilibrium in fixed income markets implies reserves and rates in the downward-sloping region for reserve demand in Figure 2. In contrast, an *equilibrium with a floor system* is such that $i^F = i^M$. In both cases, steady state output and inflation must satisfy (4), which does not depend on the banking system.

Proposition 4.3 (Steady state equilibria)

(a) For any inflation rate π, there is a target for the interbank rate i^F such that for any sufficiently small interest rate on reserves i^M, there exists a steady state equilibrium with a corridor system and inflation rate π.

(b) For any inflation rate π and any parameters $M^r > 0$ and $\mu < 1$ of the reserve supply rule (27), *there is an interest rate on reserves i^M such that there exists a steady state equilibrium with a floor system and inflation rate π.*

The key to this result is that, under either system, the government effectively controls households' steady state cost of liquidity $i^S - i^D$. This because interest rate policy and the provision of safe reserves jointly determine banks' marginal cost of producing money, that is, the average convenience yield of assets banks hold to back money on the right hand side of (23). Since the short rate is equal to the rate of time preference $i^S = \delta$ in the long run, interest rate policy sets steady state spreads on reserves and Fed funds. At the same time, the spread on other bank assets reflects those policy spreads to ensure absence of arbitrage, as required by the bank first order conditions (22). Moreover, the ratios M/D and A/D are pinned down by policy and the fact that banks' leverage constraints bind.

The result underscores the similarity between the CBDC model and the model with banks. In both cases, the policy rate that is consistent with zero inflation is not the rate of time preference but instead reflects the convenience yield on money. In the CBDC model, this follows directly because we identify the policy rate with the interest rate on money. In the model with banks here, the policy rate is a rate on short bonds that back money, so the same basic logic applies.

4.2 Linearized model

As for the economy without banks in the previous section, we study linear dynamics around a steady state with zero inflation. Equilibrium with banks is characterized by a system of linear difference equations that is very similar in structure to that for the simple model in Section 2. In fact, the nonbank private sector equations – that is, the Phillips curve and Euler equation (6) as well as household money demand (8) – continue to hold exactly as before.

What is new is that the interest rate on money as well as its real quantity are endogenous, and policy affects the market for money indirectly via the interbank loan and reserves markets. We thus replace (8) and the exogenous money supply rule by a new block of equations that describes equilibrium in the market for money. This block depends on central bank operating procedure.

Equilibrium with a floor system . The dynamics around a steady state with a floor system are given by

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1} \frac{1}{\ell} \left(i_t^S - i_t^M \right), \tag{28a}$$

$$i_t^S - \delta = i_t^M - r^M + \frac{\delta - r^M}{\eta} \left(\hat{p}_t + \hat{y}_t - \hat{d}_t \right), \tag{28b}$$

$$\hat{d}_t - \hat{p}_t = \alpha_m (\hat{m}_t - \hat{p}_t) + (1 - \alpha_m) \hat{a}_t^r,, \tag{28c}$$

$$\hat{m}_t - \hat{p}_t = \mu \left(\hat{m}_{t-1} - \hat{p}_{t-1} - \Delta \hat{p}_t \right). \tag{28d}$$

where $\alpha_m := M^r / (M^r + \rho_A A^r)$ is the steady state quality-weighted share of reserves on banks' balance sheets. Here the first equation follows from (22) and (23) using the fact that $\gamma_t = 0$, the second is derived by substituting the first into (8) and the third is the linearized worst case leverage constraint (15).

In a floor system, the central bank has two policy instruments. The interest rate on reserves affects banks' marginal cost of producing money, or the spread between the short rate and the rate on collateral assets, one of which is reserves. This spread is passed through to households with a markup, as in (28a).[13] Interest rate policy thus affects households' cost of liquidity very much like in the CBDC model above. In fact, the interest rate pass-through equation (28b) is exactly analogous to the CBDC pass-through equation (8), only the name of the policy rate has changed. The second policy instrument is the supply of reserves (28d), which directly translates into changes in the supply of money as in (28c).

Equilibrium with a corridor system . The dynamics around a steady state with a corridor

[13] Moreover, the cost channel coefficient χ defined in (7) thus depends on the policy rate and bank leverage via the steady state version of the deposit pricing equation (28a).

system are given by

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1}\frac{1}{\ell \rho_F}\left(i_t^S - i_t^F\right) \tag{29a}$$

$$i_t^S - \delta = i_t^F - r^F + \frac{\delta - r^F}{\eta}\left(\hat{p}_t + \hat{y}_t - \hat{d}_t\right), \tag{29b}$$

$$\hat{d}_t - \hat{p}_t = \frac{\eta}{\eta + \varepsilon}\hat{a}_t^r + \frac{\varepsilon}{\eta + \varepsilon}\left(\hat{y}_t - \frac{\eta}{r^F - r^M}\left(\left(i_t^F - i_t^M\right) - \left(r^F - r^M\right)\right)\right) \tag{29c}$$

Here the first equation comes from substituting for i_t^A in (23) from (26), the second follows from substituting for i^D in (8) from the first and the third follows from substituting for i_t^S in the second from (25). Since reserves are not an independent policy tool, there is no equation for their evolution.

Banks' pricing of money and pass-through from the policy rate to the short rate work qualitatively in the same way as with CBDC or a floor system. The only difference between the first two equations in (29) and (28) is that the policy rate is now the interbank rate i_t^F. The general principle is that the pricing of money depends on the cost of collateral for banks, which in turn is represented by the policy spread. The new feature with a corridor system is that banks' supply of money responds elastically to income and interest rates, as in (29c). The strength of the response depends on the distribution of liquidity shocks via the elasticity ε defined in (25). If $\varepsilon = 0$ and $i_t^F = i_t^M$ then the corridor system equations reduce to those for a floor system with a negligible supply of reserves.

How does an elastic money supply affect the transmission of interest rate policy? Suppose the central bank tightens by increasing the target for the interbank rate. As banks face higher costs of managing liquidity, they reduce the money supply so as to become constrained less often – the threshold shock λ_t^* declines. The reduction in deposits allows banks to economize on reserves, which carry a high opportunity cost. The central bank thus reduces the supply of reserves in order to implement the higher interbank rate. In fact, a decline in the threshold λ^* lowers the ratio of reserves to money and increases the ratio of other assets to money – banks become less liquid and better collateralized.[14]

Banking, CBDC and the standard model

To compare both policy regimes and models, it is helpful to substitute out for real balances

[14]Formally, the optimal threshold λ^* is determined from (22) with $\gamma_t = 0$. For the market to clear, (24) requires that the ratio M/D declines. From the definition of λ^*, A/D must increase in order for λ^* and M/D to both decline.

in the pass-through equation for the corridor system (29b) to obtain

$$i_t^S - \delta = i_t^F - r^F + \frac{\delta - r^F}{\eta + \varepsilon}\left(\hat{y}_t - \hat{a}_t^r\right) + \frac{\varepsilon}{\eta + \varepsilon}\frac{\delta - r^F}{r^F - r^M}\left(i_t^F - i_t^M - \left(r^F - r^M\right)\right).$$

For given elasticity of money demand η, the elasticity of deposit supply ε locates the corridor model on a spectrum between the model with a floor system and the standard New Keynesian model. Indeed, if ε is close to zero, the policy spread depends only on the convenience yield. In contrast, as ε becomes large, the government directly controls the short rate in the household stochastic discount factor. If moreover the corridor is of fixed size, that is, the spread $i_t^F - i_t^M$ is constant, then the second term vanishes; as a result the model converges exactly to the standard model as ε becomes large.

Consider now the relationship between the banking model and the CBDC model of Section 2. The CBDC model features a pass-through equation and hence qualitatively captures the fact that policy works through a convenience yield. As a result, it serves as a "reduced form" that approximates well the dynamics of the banking model. In particular, the banking model with a fixed size corridor can be viewed as a special case of the CBDC model with a higher elasticity of money demand $\eta + \varepsilon$ and a special money supply rule that simply fixes real balances. Moreover, the banking model with a floor system can be viewed as a special case of the CBDC model with money demand elasticity eta and a money supply rule that reflects the presence of a nontrivial quantity of reserves in bank assets.

An important quantitative difference between both models and the CBDC model is that the steady state spread between the short rate and the policy rate is smaller than that between the short rate and the interest rate on money. We thus expect the dampening mechanism from above to be weaker here, especially when utility is separable. At the same time, the strength of the cost channel, as captured by the coefficient χ, continues to reflect only the average cost of liquidity for *households* $\delta - r^D$. For the cost channel, it is not relevant how money is produced and what policy rate banks face; all that matters is the private sector cost of liquidity. As we will see in the numerical examples below, this effect can generate large differences to the standard model.

The role of bank market power. How does bank market power affect the transmission of policy? The dynamics of the model are qualitatively unchanged if market power is omitted. There is however one key change to the system of difference equations: the cost channel coefficient χ incorporates the markup via the steady state version of (23). With separable utility, this matters only for the deposit rate – there is no direct effect on the dynamics of the convenience yield on short bonds. More generally, when a cost channel is present ($\eta < \sigma$), then a larger markup increases the sensitivity of firms' marginal cost to households' cost of liquidity. It follows that market power accentuates the difference between interest rate policy in our bank model versus the standard New Keynesian model.

Determinacy of equilibrium. An equilibrium is a solution to the system of difference equations consisting of (6), and either (28) or (29). Appendix A.1 shows that Propositions 2.1 and 2.2 also hold for these systems. The general condition for determinacy is again that the long run response of the short rate to inflation is larger than one. For easier comparison, we write one condition for a general steady state policy rate r^P:

$$LR(i^S, \Delta\hat{p}) = \frac{\delta - r^P}{\eta + \varepsilon}\left(\frac{\alpha_m \mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})}\right) + \left(\phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi+\sigma^{-1})}\right)\left(1 + \frac{\delta - r^P}{\eta+\varepsilon}\frac{\varepsilon}{\delta - r^M}\right)$$
$$+ \left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\frac{\chi}{\varphi + \sigma^{-1}}\frac{\eta}{\eta+\varepsilon}\left(\phi_\pi - 1 - \phi_y\left(\frac{\varepsilon}{\delta - r^M} + \frac{\alpha_m \mu}{1-\mu}\right)\right).$$

This condition applies to the floor system equilibrium for $r^P = r^M$ and $\varepsilon = 0$ and to the corridor system equilibrium for $r^P = r^F$ and $\alpha^M = 0$.

While the forces generating stability are broadly the same as in the CBDC model above, details now depend on features of the banking system. In the separable case (the first line), stability can come from either a strong convenience yield effect or aggressive response to inflation and output. In a floor system, nominal rigidity in the supply of reserves as measured by μ strengthens the stabilizing role of the convenience yield, and more so the larger is the share of reserves on banks' balance sheet (larger α_m).[15] In a corridor system ($\varepsilon > 0$), that role is weaker. Moreover, the parameter χ defined in (7) is higher for a given policy spread, $\delta - r^P$, since it incorporates banks' markup, which strengthens the cost channel. If policy sets a peg, markups thus increase the scope for multiplicity; they further make a response to inflation more effective and a response to output less effective.

Bank assets, loan shocks and quantitative easing. A shock to the supply of other assets – say because bank borrowers become more constrained – works like a contraction of the money supply. It increases the convenience yield on money, and thereby also the convenience yield on reserves: as other assets become more scarce, reserves become more valuable as collateral to back broad money. From (28b), pass-through increases the shadow rate even if the central bank does not change the policy rate. Negative loan shocks thus generate a recession with deflation. While we have varied only the quantity of other assets A_t here, an exogenous change in their *quality* as measured by ρ_A would work in much the same way. For example, an announcement

[15] The more general point here is that any nominal rigidity can contribute to stability. Appendix A.1 actually also covers the case in which all assets are nominal and evolve according to a feedback rule analogous to (3). It serves to show that nominal rigidity in the stock of non-reserve government debt or even private nominal debt can help ensure stability, in either system.

that ratings of bank assets are worse than expected, would reduce quality-adjusted collateral supply, thereby reducing deposit supply and so increasing the convenience yield on money.

We can also use the model to think about unconventional balance sheet policies of the central bank in a floor system. Consider two examples. First, a swap of high quality reserves for other nominal assets of lower quality on bank balance sheets is described by $dA = -dM$ and hence $\hat{a}_t = -(M/A)\,\hat{m}_t$ so the change in the money supply from (28c) is

$$\hat{d}_t = \alpha_m \hat{m}_t - (1-\alpha_m)\frac{M}{A}\hat{m}_t = (1-\rho_A)\,\alpha_m \hat{m}_t.$$

The substitution of good for bad collateral thus increases the money supply and stimulates the economy, and more so if the collateral purchased by the central bank is of worse quality.

As a second example, consider a central bank purchase of assets *not* held by banks. In terms of our model, such bonds are held directly by households. The purchase of such bonds thus works mechanically like a "helicopter drop" of reserves: there is an increase in M^r not accompanied by a drop in other bank assets A^r. The central bank intervention effectively increases the collateral available to back inside money. The policy thus stimulates the economy even more than a purchase of assets held by banks. We recognize that to draw stronger conclusions here requires a more explicit model of why some assets are held within the banking system while others are not.[16] We can already see however, that even in a richer model a key determinant of the power of unconventional policy is in how it changes bank collateral assets and their convenience yield.

4.3 Numerical example

We provide a numerical example to show that deviations from the standard model as well differences between operating procedures can be potentially significant. We assume again that the central bank runs a Taylor rule with interest rate smoothing (12) with a coefficient 1.5 on inflation and .5 on the last interest rate. We also assume that other bank assets are real and constant. To compute steady states for either operating system, we need to select two new parameters: the average spread $\delta - r^P$ between the short rate and the policy rate and the markup factor that links the reserve and deposit spreads.

We assume that the rate on short bonds targeted by the central bank – regardless of operating procedure – is the same as the historical average of the US federal funds rate of 4.6% per

[16] Such a model might add additional institutions or intermediaries such as pension funds, insurance companies, or foreign central banks that value certain assets more than banks, and hence bid down their prices, making them unattractive as collateral to back inside money. The unconventional policy provides a way to circumvent a situation with endogenously segmented markets.

year. As in the calibration of the CBDC model above, we identify the short rate in households' the stochastic with the average short rate of 4.9% per year from the term structure model in Gurkaynak, Sack and Wright (2007). The average spread $\delta - r^P$ is 30 basis points. With an MZM own rate of 2.5% per year, the markup factor must then be about 8. Our exercise does not identify the extent to which the markup is due to market power as opposed to leverage or the weight on fed funds. It is plausible that ℓ and ρ_F are relatively close to one, so that χ mostly reflects market power. For this reason we work with the same value for both systems.

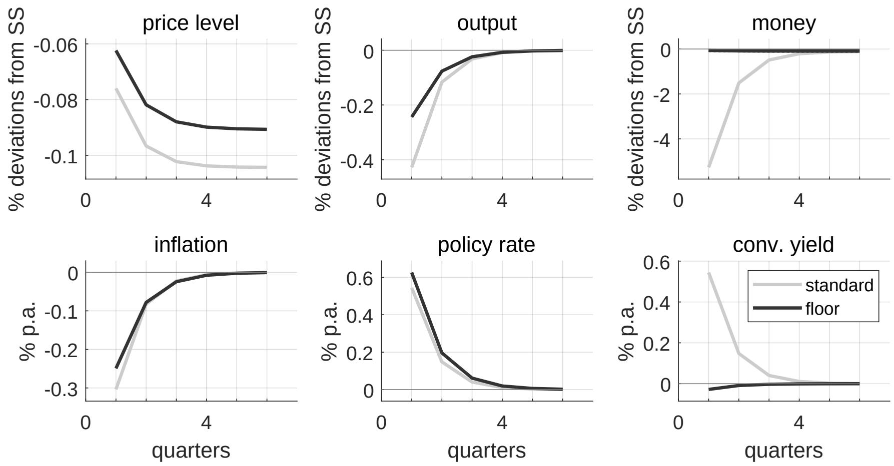

Figure 3: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate (solid lines) and difference between policy rate and deposit rate (dotted line).

Consider first the floor system. We set the share of reserves in bank assets to 15% to capture the aftermath of a sizeable QE operation. Figure 3 shows responses to a one time contractionary monetary policy shock that increases the interest rate on reserves by 25bps. The panels look essentially the same as those for the CBDC model in Figure 1. This is even though the pass-through coefficient in (28b) is much smaller in size. The reason is that the strength of the cost channel has not changed: it continues to be driven by households' cost of liquidity. The smaller policy spread is therefore not important for magnitudes. As long as the policy spread is positive, the convenience yield channel is active and the dampening effects explained above are relevant.

For the corridor system exercises, we also need values for the reserve rate and the supply elasticity ε. We set the reserve rate to zero. Banks' cost of liquidity $r^F - r^M$ is thus equal to the average policy rate of 4.6% per year. The calibration is consistent with the fact that banks' cost of liquidity was typically above households' cost of liquidity of $\delta - r^D$ of 2.4% per year in the regime with scarce reserves before 2007. The elasticity ε cannot be identified from steady state moments alone. We choose the value $\varepsilon = .24$ based on the properties of the impulse response: we require that a one percent increase in the policy rate goes along with a 50bp increase in the deposit rate. This order of magnitude is consistent with the numbers reported by Drechsler, Savov and Schnabl (2017).

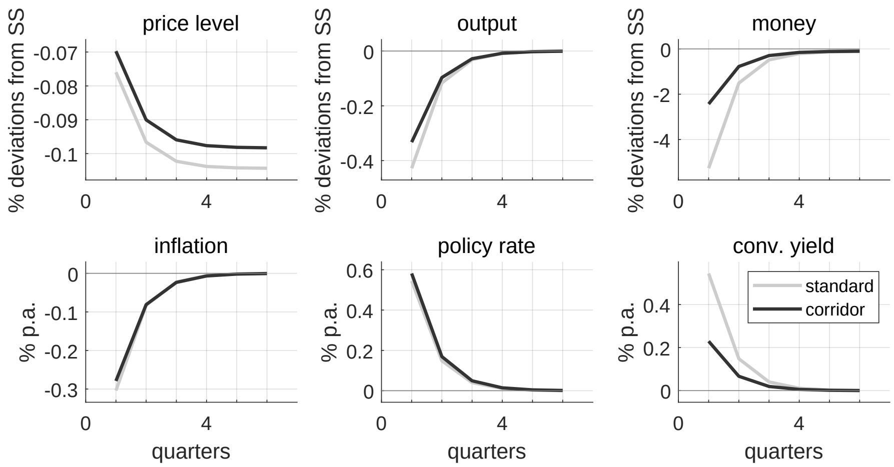

Figure 4: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate (solid lines) and difference between policy rate and deposit rate (dotted line).

Figure 4 shows responses to a one time contractionary monetary policy shock that increases the interbank rate by 25bps. Qualitatively, the shape of responses for output and inflation are now hard to distinguish from those of the standard model. Moreover, the money response is also similar as banks reduce deposits. The calibrated interest elasticity is thus high enough so as to make bank liquidity cost important. At the same time, there is still some dampening in the impulse response for output – the cost channel remains strong. The bottom left panel reports the spreads $i^S - i^D$ as a solid line as well as $i^F - i^D$ as a dashed line. Due to the small shadow spread, the two are almost identical. Calibrating to larger increases in the deposit spread – that is, more inert behavior of the deposit rate – would increase ε and drive the corridor model closer to the standard model.

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A Appendix

A.1 Characterization of equilibrium in the CBDC model

In this appendix, we collect derivations and proofs for the CBDC model of Section 2.

A.1.1 Household first-order conditions

The maximization problem of the household is:

$$\max_{\{C_t, D_t, S_t, N_t\}} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\frac{1}{\sigma}} \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \psi \frac{N_t^{1+\varphi}}{1+\varphi}$$

s.t.

$$P_t C_t + D_t + S_t \leq W_t N_t + T_t + \Pi_t + (1+i_{t-1}^D) D_{t-1} + (1+i_{t-1}^S) S_{t-1}.$$

It is helpful to introduce notation for the bundle of consumption and liquidity services consumed by the household; we define

$$B_t := \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}}.$$

Denoting the Lagrange multiplier on the budget constraint by λ_t, the household first-order conditions for consumption, money, other assets and labor are

$$\begin{aligned}
B_t^{(\frac{1}{\eta}-\frac{1}{\sigma})} C_t^{-\frac{1}{\eta}} &= \lambda_t P_t, \\
B_t^{(\frac{1}{\eta}-\frac{1}{\sigma})} \omega \left(\frac{D_t}{P_t} \right)^{-\frac{1}{\eta}} &= \lambda_t P_t - \beta(1+i_t^D) P_t E_t \left[\lambda_{t+1} \right], \\
\lambda_t &= \beta E_t \left[\lambda_{t+1} \right] (1+i_t^S), \\
\psi N_t^{\varphi} &= \lambda_t W_t.
\end{aligned}$$

To obtain a "money demand" relationship that is often studied in the empirical literature, we simplify the FOC for money by substituting for $\lambda_t P_t$ from the FOC for consumption and for $E_t \left[\lambda_{t+1} \right]$ from the FOC for other assets:

$$D_t = P_t C_t \, \omega^{\eta} \left(\frac{i_t^S - i_t^D}{1+i_t^S} \right)^{-\eta}. \tag{A.1}$$

The marginal rate of substitution of consumption for real balances must be equal to the relative price of liquidity services provided by money, or the opportunity cost of money. Since utility is homogenous of degree one in consumption and money, households hold money in proportion to nominal spending $P_t C_t$. Moreover, money holdings are decreasing in the opportunity cost of money, here the spread between other assets and money $i_t^S - i_t^D$. The elasticity of substitution η works like an interest elasticity of money demand.

The ideal price index for a bundle of consumption and liquidity services from money given by

$$Q_t := \left(1 + \omega^\eta \left(\frac{i_t^S - i_t^D}{1 + i_t^S}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{A.2}$$

This ideal price index is measured in units of consumption. Since the household cares about bundles, as opposed to only consumption goods, labor supply depends on the real wage measured in units of bundles, $W_t / P_t Q_t$. A higher spread $i_t^S - i_t^D$ thus not only increases the price of liquidity services, but also lowers the price of leisure. At the same time, it affects the household's savings decision by increasing the real return on assets in units of bundles, that is, $(1 + i_t^S) P_t Q_t / P_{t+1} Q_{t+1}$: future consumption bundles become relatively cheaper.

We can use the money demand equation to substituting out for real balances D_t / P_t in the bundle B_t

$$\begin{aligned}
B_t &= \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t}\right)^{1-\frac{1}{\eta}}\right]^{\frac{1}{1-\frac{1}{\eta}}} \\
&= \left[1 + \omega^\eta \left(\frac{i_t^S - i_t^D}{1 + i_t^S}\right)^{1-\eta}\right]^{\frac{1}{1-\frac{1}{\eta}}} C_t \\
&= Q_t^{-\eta} C_t.
\end{aligned}$$

When consumption and money are complements, an increase in the opportunity cost of money lowers the marginal utility of consumption.

The consumption FOC can now be rewritten as

$$Q_t^{\frac{\eta}{\sigma}-1} C_t^{-\frac{1}{\sigma}} = \lambda_t P_t. \tag{A.3}$$

Moreover, household labor supply (A.4) follows by combining the consumption and labor

FOCs to substitute out λ_t:

$$Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi N_t^{\varphi} = \frac{W_t}{P_t}. \tag{A.4}$$

When consumption and money are complements, an increase in the opportunity cost of money lowers labor supply relative to consumption. Indeed, the first-order conditions imply a second intratemporal Euler equation that links the marginal rate of substitution of labor for consumption to the real wage. In the separable case, the optimal choice of labor relative to consumption depends only on the relative price between these two goods: the real wage in units of consumption. When money and consumption are complements, in contrast, an increase in the opportunity cost of money makes consumption less attractive and leads households to take more leisure. Relative to the standard model, there is a "labor wedge" that is increasing in the opportunity cost of money.[17] This *cost channel* was emphasized in early flexible price DSGE models, but has received less attention in the new Keynesian literature.

Substituting out λ_t from (A.3) we arrive at an intertemporal Euler equation for the shadow rate. It relates the marginal utilities of consumption at different dates to interest rates:

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S \right) = 1. \tag{A.5}$$

Optimal savings implies that the discounted gross rate of return on assets is equal to one. In the nonseparable case, discounting by the marginal rate of substitution reflects the expected change in the opportunity cost of money. In particular, when money and consumption are complements the household acts as if he discounts the future more when the opportunity cost of money is temporarily lower: cheap liquidity today encourages consumption today.

Combining (A.1) and (A.5), we can write an analogous intertemporal Euler equation for money. It clarifies that money is valued not only for its payoff, but also earns a convenience yield:

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D \right) + \omega \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}} = 1. \tag{A.6}$$

The total return on money on the left hand side now consists not only of the pecuniary rate of

[17] With elasticities below one ($\eta < \sigma \leq 1$), competing income effects determine the labor wedge. These income effects dominate both the choice between consumption and liquidity services, and the choice between bundles and labor. A higher spread today makes liquidity services more expensive and, with a strong income effect, reduces consumption. A higher price for liquidity services also makes leisure cheaper and, with a strong income effect, increases demand for the bundle which includes more consumption. With separable utility, the two forces exactly cancel, and we obtain the Euler equation for labor from the standard model. Complementarity between money and consumption ($\eta < \sigma$) makes the income effect from the cost of liquidity stronger: a higher spread today thus leads the agent to consume relatively less and take relatively more leisure.

return (again appropriately discounted) but also adds a nonpecuniary benefit that is increasing in the velocity of money $V_t := P_t C_t / D_t$: if spending is high relative to money, shopping is more of a hassle and the convenience yield – the marginal benefit of additional money – is higher. The response of the convenience yield to velocity is stronger if the interest rate elasticity of money demand η is lower.

A.1.2 Linearization.

We follow the literature in writing log deviations from steady state in gross rates of return as deviations from steady state in net returns. For example, the gross return on money deposits is $1 + i_t^D$, and we write the log deviation from the steady state rate as

$$\log\left(1 + i_t^D\right) - \log\left(1 + i^D\right) \approx i_t^D - i^D.$$

This approximation is justified if rates of return are small, as is the case in our quarterly model with riskfree assets.

For money demand, we simplify notation by performing an additional approximation:

$$\hat{v}_t \approx \eta \frac{1 + r^D}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D) \approx \frac{\eta}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D). \tag{A.7}$$

The first equality is justified by loglinearizing and expressing rates of return in net levels, as explained above. The second equality is justified by recognizing that the small steady state return r^D multiplies small spreads $i_t^S - i_t^D$ and so we treat the product as second order.

The derivation of the New Keynesian Phillips curve and Euler equation follow the textbook treatment by Gali (2008). The Phillips curve relates the growth rate of the price level to future price growth as well as marginal cost:

$$\Delta \hat{p}_t = \beta E_t \Delta \hat{p}_{t+1} + \lambda \hat{mc}_t.$$

Since labor is the only factor of production and we abstract from the productivity shock, marginal cost variation is only variation in wages, that is, $\hat{mc}_t = \hat{w}_t$.

To find the variation in wages, consider first the effect of the cost of liquidity on the price of a bundle of consumption and liquidity. We write $Z_t = \frac{i_t^S - i_t^D}{1 + i_t^S}$ for the price of liquidity and find

$$
\begin{aligned}
\hat{q}_t &= \frac{\omega^\eta Z^{1-\eta}}{1+\omega^\eta Z^{1-\eta}} \hat{z}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \hat{z}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \eta^{-1} \hat{v}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \frac{1}{\delta - r^D} (i_t^S - i_t^D - (\delta - r^D) \\
&= \frac{\chi}{\delta - r^D} (i_t^S - i_t^D - (\delta - r^D),
\end{aligned}
$$

where the second and third line substitute for the steady state price Z and the log deviation $\hat{z}_t$, respectively, from (A.1), the fourth line substitutes for $\hat{v}_t$ from (A.7) and the fifth line defines the parameter χ: it measures the response of the price of a bundle to the price of liquidity.

The loglinearized FOC for labor is now

$$
\begin{aligned}
\hat{w}_t &= \left(1 - \frac{\eta}{\sigma}\right) \hat{q}_t + \frac{1}{\sigma} \hat{y}_t + \varphi \hat{n}_t \\
&= \left(1 - \frac{\eta}{\sigma}\right) \frac{\chi}{\delta - r^D} (i_t^S - i_t^D - (\delta - r^D) + \frac{1}{\sigma} \hat{y}_t + \varphi \hat{n}_t \\
&= \left(1 - \frac{\eta}{\sigma}\right) \frac{\chi}{\delta - r^D} (i_t^S - i_t^D - (\delta - r^D) + \frac{1}{\sigma} \hat{y}_t + \varphi \hat{n}_t,
\end{aligned}
$$

where the third line follows from the production function and the fact that we abstract from productivity shocks, so $\hat{y}_t = \hat{n}_t$. Finally, substituting wages for marginal cost, the Phillips curve takes the form in (6):

$$
\Delta \hat{p}_t = E_t \Delta \hat{p}_{t+1} + \lambda \left((\varphi + \frac{1}{\sigma}) \hat{y}_t + \left(1 - \frac{\eta}{\sigma}\right) \frac{\chi}{\delta - r^D} (i_t^S - i_t^D - (\delta - r^D)) \right).
$$

A.2 Determinacy properties

In this section we study a general system of difference equations that nests all versions of our model. After introducing notation to write the system in matrix form, we state Proposition A.1 that nests Propositions 2.1 and 2.2 in the text, and also shows that Proposition 2.2 continues to hold in the bank model of Section 4.

To set up the general system, we denote by $\hat{v}_t$ the log deviation of velocity from the steady state. We also write i_t^P for a generic policy interest rate, i_t^S for the shadow rate and $\hat{n}_t$ for exogenous nominal assets. We then consider the following system in $(\hat{y}_t, \hat{v}_t, i_t^P, i_t^S, \hat{n}_t - \hat{p}_t)$:

$$\Delta \hat{p}_t = \beta E_t \Delta \hat{p}_{t+1} + \lambda \left(\left(\varphi + \frac{1}{\sigma} \right) \hat{y}_t + \left(\frac{1}{\eta} - \frac{1}{\sigma} \right) \chi \hat{v}_t \right) \tag{A.8}$$

$$\hat{y}_t = E_t \hat{y}_{t+1} - \sigma (i_t^S - E_t \Delta \hat{p}_{t+1} - \delta) + \sigma \left(\frac{1}{\eta} - \frac{1}{\sigma} \right) \chi E_t \Delta \hat{v}_{t+1} \tag{A.9}$$

$$i_t^S - \delta = i_t^P - r^P + \frac{\delta - r^P}{\eta} \hat{v}_t \tag{A.10}$$

$$i_t^P = r^P + \phi_y \hat{y}_t + \phi_\pi \Delta \hat{p}_t + u_t \tag{A.11}$$

$$\hat{n}_t - \hat{p}_t = \mu \left(\hat{n}_{t-1} - \hat{p}_{t-1} \right) - \mu \alpha \Delta \hat{p}_t \tag{A.12}$$

$$\hat{v}_t = \frac{\eta}{\eta + \epsilon} \left(\hat{p}_t + \hat{y}_t - \hat{n}_t \right) + \frac{\eta}{\eta + \epsilon} \frac{\epsilon}{\delta - r^M} \left(i_t^P - r^P \right) \tag{A.13}$$

We are interested in bounded solutions given some initial condition for the real value of nominal assets $\hat{n}_{-1} - \hat{p}_{-1}$.

All equilibria characterized in the paper can be reduced to special cases of this system. While equilibria also describe other endogenous variables such as the deposit rate or the interest rate on other assets, those variables are simple functions of $i_t^S, \hat{p}_t$ and $\hat{y}_t$ that are not important for characterizing determinacy. The first two equations (A.8) and (A.9) are derived from household and firm behavior and hence hold in all equilibria. They follow from (6)) and (8) by substituting out the deposit rate i^D.

Equations (A.10)-(A.13) differ across models of types of equilibria some of the coefficients as well as in what interest rate represents the policy rate and what quantity represents exogenous nominal assets (if any). In particular, the system of difference equations in the CBDC model

, given by (9b), is a special case of the system (A.8) - (A.13), where the policy rate is the deposit rate $i_t^P = i_t^D$, nominal assets are deposits $\hat{n}_t = \hat{d}_t$, and we have $\alpha = 1$ and $\epsilon = 0$.

The system of difference equations for the model with banks from Section 3 depends on the type of equilibrium. For an equilibrium with a floor system, given by (6), (28b) and (28c), is a special case with the policy rate is the reserve rate $i_t^P = i_t^M$, nominal assets are deposits $\hat{n}_t = \hat{d}_t$, and we have $\alpha = \alpha_m$ and $\epsilon = 0$. Finally, the system of difference equations for the model with a corridor system from Section 4, given by (6), (9a) and (29) is the special case where the policy rate is the interbank rate $i_t^P = i_t^F$, and there are no nominal assets so $\mu = 0$ and α is irrelevant.

Substituting out for velocity $\hat{v}_t$ and the two interest rates, we have a three equation system for inflation, the real value of nominal assets, and output. In matrix notation, it is

$$\begin{bmatrix} E_t \Delta p_{t+1} \\ E_t \hat{y}_{t+1} \\ \hat{n}_t - \hat{p}_t \end{bmatrix} = A \begin{bmatrix} \Delta p_t \\ \hat{y}_t \\ \hat{n}_{t-1} - \hat{p}_{t-1} \end{bmatrix} + b_t \tag{A.14}$$

with initial condition $\hat{n}_{-1} - \hat{p}_{-1}$ and where b_t is a vector of exogenous variables.

To ease notation, we define the non-negative coefficients

$$B = \left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\chi, \quad \gamma = \delta - r^P, \quad \kappa = \lambda\left(\varphi + \frac{1}{\sigma}\right),$$
$$A_V = \frac{\eta}{\eta + \epsilon}, \quad B_V = A_V \frac{\epsilon}{\delta - r^M}, \quad \Gamma = \beta\left(\sigma^{-1} + BA_V + BB_V\phi_y\right).$$

We write A_{ij} for the element in the ith row and jth column of A. We thus have $A_{31} = -\mu\alpha$, $A_{32} = 0$, $A_{33} = \mu$, and the elements in the first two rows are:

$$A_{11} = \left(\frac{1 - \mu\alpha A_V \lambda B - \lambda B B_V \phi_\pi}{\beta}\right), A_{12} = -\left(\frac{\kappa + B\lambda A_V + B\lambda B_V \phi_y}{\beta}\right), A_{13} = \frac{\mu \tilde{B} A_V}{\beta},$$
$$A_{21} = \frac{\beta}{\Gamma}\left[\phi_\pi + BB_V\phi_\pi + \frac{\gamma\mu\alpha}{\eta}A_V + \frac{\gamma}{\eta}B_V\phi_\pi + \mu\alpha(1-\mu)BA_V\right],$$
$$-\frac{\beta}{\Gamma}\left[(1 + \mu\alpha BA_V + BB_V\phi_\pi)\left(\frac{1 - \alpha\mu A_V B\lambda - B\lambda B_V\phi_\pi}{\beta}\right)\right],$$
$$A_{22} = 1 + \frac{\beta}{\Gamma}\left[\phi_y + \frac{\gamma}{\eta}A_V + \frac{\gamma}{\eta}B_V\phi_y + (1 + \mu\alpha BA_V + BB_V\phi_\pi)\left(\frac{\kappa + A_V B\lambda + B\lambda B_V\phi_y}{\beta}\right)\right],$$
$$A_{23} = -\frac{\beta}{\Gamma}\left[BA_V\mu(1-\mu) + \frac{\mu\gamma}{\eta}A_V + (1 + \mu\alpha BA_V + BB_V\phi_\pi)\left(\frac{\mu B\lambda A_V}{\beta}\right)\right].$$

To state the proposition, we define the long run responses to a change in inflation. From the law of motion for the real value of nominal assets, we have

$$LR(\hat{n} - \hat{p}, \Delta\hat{p}) = -\frac{\mu}{1-\mu}\Delta\hat{p}$$

From the Phillips curve, the response of output to inflation is

$$LR(\hat{y}, \Delta\hat{p}) = \left(\frac{(1-\beta) - \tilde{B}B_V\phi_\pi - \tilde{B}A_V\left(\frac{\alpha\mu}{1-\mu}\right)}{\kappa + \tilde{B}A_V + \tilde{B}B_V\phi_y}\right)\Delta\hat{p}$$

Finally, using the Taylor rule, (A.11), and the pass-through equation, (A.10), we have that:

$$
\begin{aligned}
LR\left(i^S, \Delta\hat{p}\right) &= LR(i^P, \Delta\hat{p}) + LR(i^S - i^P, \Delta\hat{p}) \\
&= \left(\phi_\pi + A_V \frac{\gamma}{\eta}\frac{\alpha\mu}{1-\mu} + \frac{\gamma}{\eta} B_V \phi_\pi\right)\Delta\hat{p} + \left(\phi_y + \frac{\gamma}{\eta}(A_V + B_V\phi_y)\right) LR(\hat{y}, \Delta\hat{p}).
\end{aligned}
$$

We impose throughout (11), written in the notation here as

Condition 1: $\phi_y B\lambda < \kappa\gamma/\eta$.

Proposition A: Suppose Condition 1 holds. The system of difference equations (A.8) - (A.13) has a unique bounded solution for any initial condition $(\hat{n}_{-1} - \hat{p}_{-1})$ *if and only if*

$$
\frac{LR(i^S, \Delta\hat{p})}{\Delta\hat{p}} > 1. \tag{A.15}
$$

Proof. We show that the matrix A in (A.14) has exactly one eigenvalue inside the unit circle if and only if (A.15) holds. We further check the rank condition on A in Blanchard and Kahn (1980). It then follows that, (A.15) guarantees a unique bounded solution to (A.14).

The characteristic polynomial of A

The eigenvalues of A are the roots of its characteristic polynomial

$$
p(\lambda) = \lambda^3 - a_2\lambda^2 + a_1\lambda - a_0
$$

where the coefficients take the form

$$
\begin{aligned}
a_2 = &\left(1 + \frac{1}{\beta} + \mu\right) + \left(\frac{1}{\Gamma}\right)\left[\beta\frac{\gamma}{\eta}A_V + \kappa + B\lambda A_V(1 + \mu\alpha\phi)\right] \\
&+ \left(\frac{\phi_y}{\Gamma}\right)\left[\beta(1 + \frac{\gamma}{\eta}B_V) + B\lambda B_V\right] + \left(\frac{\phi_\pi}{\Gamma}\right) B\lambda B_V \phi > 2 \qquad \text{(A.16)} \\
a_1 = &\frac{1 + \mu + \mu\beta}{\beta} + \left(\frac{1}{\Gamma}\right)\left[(1 + \mu\beta)\frac{\gamma}{\eta}A_V + \mu\kappa(1 + \frac{\alpha\gamma}{\eta}A_V) + \mu A_V B\lambda(1 + \alpha\phi)\right] \\
&+ \left(\frac{\phi_y}{\Gamma}\right)\left[(1 + \mu\beta)(1 + \frac{\gamma}{\eta}B_V) - \alpha\mu A_V B\lambda + \mu B\lambda B_V\right] \\
&+ \left(\frac{\phi_\pi}{\Gamma}\right)\left[\kappa + \kappa\frac{\gamma}{\eta}B_V + B\lambda B_V(1 + \mu)\phi + B\lambda A_V\right] \qquad \text{(A.17)} \\
a_0 = &\frac{\mu}{\beta} + \frac{\mu}{\Gamma}\left[\frac{\gamma}{\eta}A_V\right] + \frac{\mu\phi_y}{\Gamma}\left[1 + \frac{\gamma}{\eta}B_V\right] + \frac{\mu\phi_\pi}{\Gamma}\left[\kappa(1 + \frac{\gamma}{\eta}B_V) + B\lambda A_V + B\lambda B_V\phi\right] \qquad \text{(A.18)}
\end{aligned}
$$

We note that $a_2 > 2$ and $a_0 \geq 0$, with strict inequality if and only if $\mu > 0$. Moreover, Condition 1 implies that $a_1 > 1$. The characteristic polynomial thus has a root at zero if and only if $\mu = 0$. For $\mu > 0$, Descartes' rule of signs implies that the polynomial has either one or three positive real roots and no negative real roots. We thus always have one positive real root. In addition, there could be two more positive real roots, or there could be a pair of complex conjugates.

We have $p(0) = -a_0$ and the values of the polynomial at plus and minus one are:

$$
\begin{aligned}
p(-1) = & -\frac{2}{\beta}(1+\mu)(1+\beta) - \left(\frac{1}{\Gamma}\right)\left[\frac{\gamma}{\eta}A_V((1+\mu)(1+\beta)+\mu\kappa) + \kappa(1+\mu) + B\lambda A_V(1+\mu+2\alpha\mu\phi)\right] \\
& - \left(\frac{\phi_y}{\Gamma}\right)\left[(1+\frac{\gamma}{\eta}B_V)(1+\mu)(1+\beta) + B\lambda B_V(1+\mu)\right] \\
& - \left(\frac{\phi_\pi}{\Gamma}\right)\left[(1+\mu)(\kappa + B\lambda A_V) + \kappa B_V\frac{\gamma}{\eta}(1+\mu) + B\lambda B_V\phi(2+2\mu)\right] - \left(\frac{1}{\Gamma}\right)\alpha\mu A_V\left(\frac{\kappa\gamma}{\eta} - \phi_y B\lambda\right)
\end{aligned}
$$

$$
\begin{aligned}
p(1) = & \left(\frac{1}{\Gamma}\right)\left[(1-\mu)(\kappa + B\lambda A_V + B_V\kappa\frac{\gamma}{\eta})(\phi_\pi - 1) + (1-\mu)(1-\beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1))\right] \\
& + \left(\frac{1}{\Gamma}\right)\left[(\alpha\mu A_V + (1-\mu)B_V)(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right]
\end{aligned}
$$

Condition (A.15) is equivalent to $p(1) > 0$. It further implies that $p(-1) < 0$ for all μ.

Necessity of A.15

We now establish necessity of (A.15). If the difference equation has a unique bounded solution, exactly one eigenvalue of A is inside the unit circle. In other words, exactly one root of $p(\lambda)$ is inside the unit circle. We know that $p(-1) < 0$. If $p(1) \leq 0$, then the polynomial crosses the horizontal axis within $(-1, 1)$ either twice or never. Thus, it is necessary that $p(1) > 0$ i.e. (A.15) holds.

To establish sufficiency, we first show that (A.15) ensures a unique stable eigenvalue, that is, the condition $p(1) > 0$ ensures exactly one root of $p(\lambda)$ lies inside the unit circle. It is convenient to do this part of the proof in two steps, first for $\mu = 0$ and then for $\mu \in (0, 1]$.

(A.15) *implies a unique stable eigenvalue for $\mu = 0$.*

The case $\mu = 0$ is special because $a_0 = 0$. The three roots $(\lambda_1, \lambda_2, \lambda_3)$ are

$$\lambda_1 = 0; \lambda_2 = \frac{a_2 - \sqrt{a_2^2 - 4a_1}}{2}; \lambda_3 = \frac{a_2 + \sqrt{a_2^2 - 4a_1}}{2}$$

The roots λ_2, λ_3 are either both real or both complex. If they are both complex or they are real and equal, then both must lie outside the unit circle since $a_2 > 2$. Assume instead they are both real and distinct, with $\lambda_2 < \lambda_3$. We know that $\lambda_2 > 1$ if and only if:

$$\begin{aligned} a_2^2 - 4a_1 &< a_2^2 - 4a_2 + 4 \\ \text{i.e. } 1 - a_2 + a_1 &> 0 \end{aligned}$$

If $\mu = 0$, (A.15) can be written as $p(1) = 1 - a_2 + a_1 > 0$. We therefore have $\lambda_2 > 1$ and hence also $\lambda_3 > 1$. It follows that exactly one root, $\lambda = 0$, lies within the unit circle.

(A.15) *implies a unique stable eigenvalue for* $\mu > 0$.

We show next that for any $\mu \in (0, 1]$, (A.15) also ensures that there is a unique root of $p(\lambda)$ inside the unit circle. We know that there can be either one or three roots inside the unit circle. Indeed, $p(1) > 0$, given by (A.15), and $p(0) < 0$ imply that the polynomial has either one or three real roots in the interval $(0, 1)$. Moreover, if there is a pair of complex roots, those roots have the same modulus. We thus want to rule out that there are three roots inside the unit circle. The following result provides restrictions on a cubic polynomial that allows this case:

Lemma A1: Suppose $p(\lambda) = \lambda^3 - b_2\lambda^2 + b_1\lambda - b_0$ *is a cubic polynomial with strictly positive real-valued coefficients* b_2, b_1, b_0 *that satisfies* $p(1) > 0$*. If all roots lie within the unit circle, then the coefficients satisfy*

(a) $b_0 < 1$,

(b) $b_0^2 - b_2 b_0 + b_1 - 1 < 0$,

(c) $b_2 < 2 + b_0$.

Proof.

Part (a): Denote the three roots by $(\lambda_1, \lambda_2, \lambda_3)$, where λ_1 is the smallest real root, and (λ_2, λ_3) are either both real roots or both complex roots. Since we can write the polynomial as $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, we have $b_0 = \lambda_1\lambda_2\lambda_3$. Suppose (λ_2, λ_3) are real roots. We know they must both be positive. Since $p(0) < 0$ and $p(1) > 0$, we have $\lambda_2\lambda_3 \in (0, 1)$ and hence $b_0 < \lambda_1 < 1$. If instead (λ_2, λ_3) are complex roots, then $\lambda_2\lambda_3 = |\lambda_2|^2 = |\lambda_3|^2 \in (0,1)$ and again $b_0 < \lambda_1 < 1$.

Part (b): We have

$$\begin{aligned} p(b_0) &= b_0^3 - b_2 b_0^2 + b_1 b_0 - b_0 \\ &= b_0 \left(b_0^2 - b_2 b_0 + b_1 - 1 \right). \end{aligned}$$

We show that $p(b_0) < 0$. Since $b_0 > 0$, Condition (b) then follows.

To show $p(b_0) < 0$, suppose first that (λ_2, λ_3) are both real roots. then both turning points of $p(\lambda)$ must be larger than λ_1. It follows that $p(\lambda) < 0$ for any $\lambda < \lambda_1$. From the proof of part (a), we have that $b_0 < \lambda_1$ and hence $p(b_0) < 0$. If instead (λ_2, λ_3) are both complex roots, $p(\lambda)$ only crosses the horizontal axis once at $\lambda = \lambda_1$. Since $p(0) < 0$, then $p(\lambda) < 0$ for any $\lambda \in (0, \lambda_1)$. As $b_0 < \lambda_1$, again $p(b_0) < 0$.

Part (c): We start from Condition (b) and use our assumption that $p(1) = 1 - b_2 + b_1 - b_0 > 0$ to obtain

$$\begin{aligned} 0 > b_0^2 - b_2 a_0 + b_1 - 1 > b_0^2 - b_2 a_0 + (b_2 + b_0 - 1) - 1 \\ = b_0^2 - (b_2 - 1) a_0 + (b_2 - 2) \\ = (1 - b_0)(b_2 - 2 - b_0). \end{aligned}$$

Condition (c) follows because Condition (a) ensures that $b_0 < 1$.■

We now show that Condition 1 does not allow Conditions (a)-(c) of Lemma A1 to hold jointly for our characteristic polynomial. It then follows that we cannot have three roots inside the unit circle, and thus have exactly one root inside the unit circle. We first note that there exists a threshold value $\bar{\mu} < \beta < 1$ such that Condition (a) of Lemma A1 is violated for all $\mu > \bar{\mu}$. Indeed, we can always find $\bar{\mu}$ such that $a_0 = 1$. For the remaining case $\mu < \bar{\mu}$, we have the following Lemma:

Lemma A2: Assume that Condition 1 holds. Suppose the characteristic polynomial $p(\lambda) = \lambda^3 - a_2\lambda^2 + a_1\lambda - a_0$ with coefficients (A.16) - (A.18) *satisfies Conditions (a) and (c) of Lemma A1. Then Condition (b) of Lemma A1 does not hold.*

Proof. Condition (c) of Lemma A1 applied to our characteristic polynomial is given by

$$
\begin{aligned}
&-(1/\beta-1)(1-\mu)-\frac{1}{\Gamma}\left[(\beta-\mu)(\frac{\gamma}{\eta}A_V+\phi_y(1+\frac{\gamma}{\eta}B_V))+B\lambda A_V\alpha\mu\phi+\phi_\pi(1-\mu)B\lambda B_V\phi\right] \\
&+\frac{1}{\Gamma}\left[(\kappa+B\lambda A_V+\kappa\frac{\gamma}{\eta}B_V)(\mu\phi_\pi-1)+B_V(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda)\right]>0. \qquad \text{(A.19)}
\end{aligned}
$$

To check Condition (b) of Lemma A1, we define the function $g(\mu) := a_0^2 - a_2 a_0 + a_1 - 1$, where dependence of the coefficients on μ is given by (A.16) - (A.18). In particular, the coefficients are linear in μ, so the function $g(\mu)$ is quadratic in μ. We want to show that $g(\mu) > 0$ for all $\mu \in (0, \bar{\mu}]$, thus violating Condition (b).

We know from (A.16) - (A.18) that $g(0) = a_1 - 1 > 0$. Since Condition (c) of Lemma A1 is assumed to hold, we also know

$$
\begin{aligned}
g(\bar{\mu}) &= a_1 - a_2 \\
&= \frac{1}{\Gamma}(1-\bar{\mu})\left[(\kappa+B\lambda A_V+\kappa\frac{\gamma}{\eta}B_V)(\phi_\pi-1)+B_V(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda)\right] \\
&+\frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V+\phi_y(\frac{\gamma}{\eta}B_V+1))+\alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda)\right] \\
&> \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V+\phi_y(\frac{\gamma}{\eta}B_V+1))+\alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda)\right] \\
&> 0,
\end{aligned}
$$

where the third line uses (A.19) and $\bar{\mu} < \beta$, and the last line follows from Condition 1.

It remains to show that the function $g(\mu)$ is also positive in the interior of the interval $[0, \bar{\mu}]$. Since $g(\mu)$ is quadratic, it is either concave or convex everywhere. If it is concave, then $g(0) > 0$ and $g(\bar{\mu}) > 0$ imply that $g(\mu)$ is positive over the entire interval $[0, \bar{\mu}]$. Suppose therefore that $g(\mu)$ is convex. If the derivative of $g(\mu)$ at $\bar{\mu}$ is negative, then $g(\mu) > g(\bar{\mu}) > 0$ for all $\mu < \bar{\mu}$. If instead the derivative of $g(\mu)$ at $\bar{\mu}$ is positive, then $g(\mu)$ is bounded below by the function

$$h(\mu) := g(\bar{\mu}) + (\mu - \bar{\mu})g'(\bar{\mu}).$$

We proceed to show that $h(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$, hence $g(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$. As $g'(\bar{\mu}) > 0$, then $h'(\mu) > 0$, implying that if $h(0) > 0$, then $h(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$. This is what we show.

The derivative of $g(\mu)$ at the point $\bar{\mu}$ is

$$
\begin{aligned}
g'(\bar{\mu}) = & -(\frac{1}{\beta}-1)(\frac{1}{\bar{\mu}}-1) \\
& -\left(\frac{1}{\Gamma}\right)\left(\frac{1}{\bar{\mu}}-1\right)\left(\beta\frac{\gamma}{\eta}A_V+\kappa+B\lambda A_V+\phi_y(\beta(1+\frac{\gamma}{\eta}B_V)+B\lambda B_V)+\phi_\pi B\lambda B_V\phi\right) \\
& -\left(\frac{1}{\Gamma}\right)\left(\alpha\phi_y A_V B\lambda-\alpha\kappa\frac{\gamma}{\eta}A_V+\alpha B\lambda A_V\phi\right) \\
& < \left(\frac{\alpha}{\Gamma}\right)A_V\left(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda\right).
\end{aligned} \tag{A.20}
$$

Substituting into the definition of h, we have that:

$$
\begin{aligned}
h(0) &= g(\bar{\mu})-\bar{\mu}g'(\bar{\mu}) \\
&> g(\bar{\mu})-\frac{\alpha\bar{\mu}}{\Gamma}A_V\left(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda\right) \\
&> \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V+\phi_y(\frac{\gamma}{\eta}B_V+1))+\alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda)\right]-\frac{\alpha\bar{\mu}}{\Gamma}A_V\left(\kappa\frac{\gamma}{\eta}-\phi_y B\lambda\right) \\
&= \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V+\phi_y(\frac{\gamma}{\eta}B_V+1))\right] \qquad > 0,
\end{aligned}
$$

where the second line uses the bound from (A.20).■

Blanchard-Kahn Rank Condition

We have shown that (A.15) implies that the matrix A exhibits exactly one eigenvalue inside the unit circle. By Blanchard and Kahn (1980), this implies a unique bounded solution to (A.14) as long as a Rank Condition is satisfied. To check this rank condition, let B denote the matrix of left eigenvectors of A, sorted by their modulus in ascending order. We want to show that the block corresponding to the predetermined variables is nonsingular. In our context, this means showing that the top left element of B is different from zero.

Suppose this were not true, that is, we have a left eigenvector $(0, x, y)$ of A that satisfies:

$$
\begin{bmatrix} 0 & x & y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 & x & y \end{bmatrix},
$$

where λ_1 is the unique eigenvalue in $(0, 1)$. Consider the second column of the equation. Since $A_{23} = 0$, it reads $xA_{22} = \lambda_1$. It cannot hold since $A_{22} > 1$ and $\lambda_1 < 1$. ■

A.3 Characterizing partial equilibrium in fixed income markets

In this appendix we collect derivations needed to characterize partial equilibria in fixed income markets as discussed in Section 3, as well as the proofs of Propositions 3.1 and 3.2. For easier notation we drop superscripts indicating individual banks.

A.3.1 Bank market power and deposit demand

In the setup with monopolistic competition, bank i supplies liquidity to households at the price $Z_t^i = (i_t^S - i_t^{D,i})/(1+i_t^S)$, where $i_t^{D,i}$ is the deposit rate promised by bank i. The spread $i_t^S - i_t^{D,i}$ is interest foregone by investing in deposits as opposed to the shadow rate, discounted by $(1+i_t^S)$ as the interest is received next period.

Households value different varieties of deposits according to a CES aggregator with elasticity of substitution η_b. For given individual bank deposit rates $i_t^{D,i}$ and hence liquidity prices Z_t^i, let Z_t denote the ideal CES price index that aggregates the individual bank liquidity prices Z_t^i. We then define the ideal average deposit rate i_t^D by

$$\frac{i_t^S - i_t^D}{1+i_t^S} = Z_t.$$

Household maximization delivers bank i's deposit demand function

$$D_t^i = \left(\frac{Z_t^i}{Z_t}\right)^{-\eta_b} D_t = \left(\frac{i_t^S - i_t^{D,i}}{i_t^S - i_t^D}\right)^{-\eta_b} D_t. \tag{A.21}$$

Writing γ_t for the multiplier on the leverage constraint, the terms in the Lagrangian involving the date t deposit rate are

$$D_t^i - \frac{D_t^i}{1+i_t^S}\left(1+i_t^{D,i}\right) - \gamma_t D_t^i = \left(Z_t^i - \gamma_t\right) D_t^i.$$

Shareholder maximization thus works like profit maximization with constant marginal cost γ_t via choice of a price Z_t^i.

The first order conditions with respect to Z_t^i take the standard form

$$\left(Z_t^i - \gamma_t\right)\eta_b \left(Z_t^i\right)^{-\eta_b - 1}\frac{D_t}{Z_t^{-\eta_b}} + \left(\frac{Z_t^i}{Z_t}\right)^{-\eta_b} D_t = 0,$$

$$\eta_b \left(\frac{i_t^S - i_t^{D,i}}{i_t^S - i_t^D}\right)^{-\eta_b - 1} \frac{1}{i_t^S - i_t^D} \left(1 - \frac{1 + i_t^{D,i}}{1 + i_t^S} - \gamma_t\right) D_t - \frac{1}{1 + i_t^S} D_t^i = 0.$$

A higher price of liquidity lowers profits by decreasing the quantity of deposits, but increases profits by increasing revenue per dollar issued.

Substituting from the demand function and rearranging, we have

$$i_t^S - i_t^{D,i} = \frac{\eta_b}{\eta_b - 1} \left(1 + i_t^S\right) \gamma_t.$$

Bank i chooses a price that multiplies marginal cost by a constant markup.

Combining the reserves and deposits first order condition, we arrive at equation (23):

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1} \ell^{-1} \left(i_t^S - i_t^M\right).$$

A.3.2 Proof of Proposition 3.1

A bank's problem in the second subperiod is to choose M', F^+ and F^- to maximize next period cash

$$M' \left(1 + i^M\right) + \left(1 + i^F\right) \left(F^+ - F^-\right),$$

subject to the budget and collateral constraints as well as nonnegativity constraints on all three variables.

Assume first that $i^F > i^M$. The first order conditions are

$$\begin{aligned} 1 + i^M + \gamma \ell &= \lambda - \nu_{M'}, \\ 1 + i^F + \gamma \rho_F \ell &= \lambda - \nu_{F^+}, \\ 1 + i^F + \gamma &= \lambda + \nu_{F^-}, \end{aligned}$$

where γ is the multiplier on the collateral constraint, λ is the multiplier on the budget constraint, and the νs are the multipliers on the three nonnegativity constraints.

We distinguish solutions with positive reserve holdings from those with zero reserves. Suppose first a bank holds no reserves overnight, that is, $M' = 0$. The optimal policy is then

$$F^+ - F^- = M - \lambda D$$

In order for the collateral constraint to be satisfied, we must have $D - M < \ell \rho_A A$. The precise split into F^+ and F^- is not important in this case – only the net position is determinate.

Suppose instead a bank holds reserves overnight, that is, $M' > 0$ and hence $\nu_{M'} = 0$. We must have $\gamma > 0$: otherwise the fed funds lending and reserves FOC cannot jointly hold. Indeed, these FOC imply

$$1 + i^F + \gamma \rho_F \ell \leq \lambda = 1 + i^M + \gamma \ell,$$

which cannot hold for $\gamma > 0$ since we have assumed $i^F > i^M$. From the fed funds borrowing FOC, we must then have $\nu_{F-} > 0$ and hence $F^- = 0$.

When the bank holds reserves, we can thus combine the binding collateral constraint and the budget constraint to find optimal reserve holdings and fed funds lending

$$\begin{aligned}
M' &= \frac{\left(1 - \tilde{\lambda}\left(1 - \rho_F \ell\right)\right) D - \rho_F \ell M - \rho_A \ell A}{\ell \left(1 - \rho_F\right)}, \\
F^+ &= M - \tilde{\lambda} D - M' \\
&= \frac{\left(M - \tilde{\lambda} D\right) \ell \left(1 - \rho_F\right) - \left(1 - \tilde{\lambda}\left(1 - \rho_F \ell\right)\right) D + \rho_F \ell M + \rho_A \ell A}{\ell \left(1 - \rho_F\right)} \\
&= \frac{M\ell + \rho_A \ell A - (1 - \tilde{\lambda}\left(1 - \ell\right))D}{\ell \left(1 - \rho_F\right)}
\end{aligned}$$

We need for this case that M' is positive and F^+ is nonnegative. The first condition is equivalent to $\tilde{\lambda} < \lambda^*$. The second condition is satisfied at any value of $\tilde{\lambda}$ as long as it is satisfied at $\tilde{\lambda} = -\bar{\lambda}$. The condition assumed in the proposition says that the second condition is indeed satisfied at $\tilde{\lambda} = -\bar{\lambda}$.

Finally, consider the case where $i^F = i^M$. The bank is indifferent between fed funds positions and reserves, and any feasible plan is optimal. We are interested in the lowest feasible reserve holdings. all plans that achieve this lower bound must have $F^- = 0$: if fed funds borrowing were positive, we could reduce it together with reserves and still satisfy the budget and collateral constraints. Moreover, the plans must make the collateral constraint bind. It follows that they are described by the equations for $\tilde{M}$ and F^+ above.■

A.3.3 Proof of Proposition 3.2

To save space, we drop time subscripts and superscripts for individual banks throughout this proof. An equilibrium consists of ratios M/D and A/D, a threshold liquidity shock λ^*, a multiplier on the worst case collateral constraint γ as well as interest rates i^A and i^D that satisfy the definition of the threshold shock (18), banks' first order conditions for reserves and other assets (22), the deposit pricing condition (23) and the reserve market clearing condition (24). Proposition 4.1 implies that the latter condition holds with equality if $i^F > i^M$, but may hold as a strict inequality otherwise. In addition, equilibrium ratios M/D and A/D must satisfy (15).

We distinguish *elastic equilibria* such that the worst case leverage constraint (15) is slack and $\gamma = 0$ from *inelastic equilibria* such that the worst case leverage constraint binds. We also note that the deposit rate i^D only enters the deposit pricing condition (23). In particular, we can always find a deposit rate smaller than i^S provided that $i^A < i^S$. We therefore focus on determining ratios M/D and A/D together with λ^* and i^A from (18), (22) and (24).

Parts (a) and (b) . Suppose that $i^F > i^M$. The market clearing condition for reserves (24) therefore holds with equality.

The proof proceeds in two steps that consider elastic and inelastic equilibria, respectively. Step 1 shows that there is a threshold interbank rate i_t^{F*} such that an elastic equilibrium exists for all $i^F \geq i^{F*}$, and that there is no elastic equilibrium for lower values of i^F. Step 2 shows that there exists an inelastic equilibrium for all $i^F \geq i^{F*}$, and that there is no inelastic equilibrium for higher values of i^F. Together these statements imply parts (a) and (b) of the proposition.

Step 1. We first establish that an elastic equilibrium exists only if $i^F \in \left[(1-\rho_F)\, i^S + \rho_F r^M, i^S\right]$. Indeed, in an elastic equilibrium, the threshold liquidity shock $\lambda^* \in [-\bar{\lambda}, \lambda]$ must solve the bank first order condition for reserves (22a) for $\gamma = 0$. Such a solution exists if and only if we can find a number $G(\lambda^*$ between zero and one so that (22a) holds. This is because (i) the threshold shock λ^* enters the equation only as an argument of G, and (ii) G is strictly increasing, so that there is a one-to-one mapping between $G(\lambda^*)$ and λ^*. Rearranging (22a), $G(\lambda^*) \in [0,1]$ is equivalent to

$$0 \leq \frac{1-\rho_F}{\rho_F} \frac{i^S - i^F}{i^F - i^M} \leq 1$$

This condition is in turn is equivalent to $i^F \in \left[(1-\rho_F)\, i^S + \rho_F r^M, i^S\right]$. Elastic equilibria exist only if i^F is in this range and we focus on this range from now on.

We now determine all equilibrium objects for a given candidate interbank interest rate i^F. The first order condition (22a) implies a one-to-one mapping between the threshold liquidity shock and the interbank interest rate. For future reference, we define

$$f\left(i^F\right) := G^{-1}\left(\frac{1-\rho_F}{\rho_F} \frac{i^S - i^F}{i^F - i^M}\right) \tag{A.22}$$

over the interval $\left[(1-\rho_F)\, i^S + \rho_F i^M, i^S\right]$. The function f is strictly decreasing and we have that $f\left((1-\rho_F)\, i^S + \rho_F i^M\right) = \bar{\lambda}$ and $f\left(ß^S\right) = -\bar{\lambda}$.

Given a threshold shock $\lambda^* = f\left(i^F\right)$, the equilibrium reserve ratio M/D follows from the market clearing condition (24); it is always nonnegative. The interest rate on other assets follows as the solution to the bank first order condition for other assets (22b) with $\gamma = 0$. This solution is always below i^S since the term in braces is nonnegative. Finally, the equilibrium ratio of other assets A/D is determined by the definition of the threshold shock (18).

The ratio A/D is nonnegative provided that $\bar{\lambda}$ is sufficiently small. Indeed, for any λ^*, the solution A/D to (18) is nonnegative if and only if

$$\rho_F \frac{1-\rho_F \ell}{1-\rho_F} \int_{-\bar{\lambda}}^{\lambda^*} \left(\lambda^* - \tilde{\lambda}\right) dG\left(\tilde{\lambda}\right) + \lambda^* \left(1 - \rho_F \ell\right) < 1 \tag{A.23}$$

Since $G(\lambda^*) \leq 1$, we have $\lambda^* \leq \bar{\lambda}$, this condition holds if $\bar{\lambda}$ is sufficiently small, as we assume in the proposition.

We have now constructed all equilibrium objects given a candidate interbank interest rate i_t^F. To establish existence, it remains to show for which interest rates the ratios M/D and A/D also satisfy the worst case leverage constraint (15). Conveniently, those ratios enter the constraint only in the form of the threshold shock λ^*. To see this, express the weighted sum of collateral ratios in (15) in terms of λ^* as

$$\begin{aligned}
\ell \frac{M}{D} + \rho_A \ell \frac{A}{D} &= 1 - \left(1 - \rho_F \ell\right) \lambda^* + \left(1 - \rho_F\right) \ell \frac{M}{D} \\
&= 1 - \left(1 - \rho_F \ell\right) \lambda^* + \left(1 - \rho_F \ell\right) \left(\lambda^* G\left(\lambda^*\right) - \int_{-\bar{\lambda}}^{\lambda^*} \tilde{\lambda} dG\left(\tilde{\lambda}\right)\right) \\
&= 1 - \left(1 - \rho_F \ell\right) \left(\lambda^* \left(1 - G\left(\lambda^*\right)\right) + \int_{-\bar{\lambda}}^{\lambda^*} \tilde{\lambda} dG\left(\tilde{\lambda}\right)\right) \\
&=: h\left(\lambda^*\right)
\end{aligned}$$

where the first equality uses the definition of the threshold shock λ^* and the second uses interbank market clearing (24).

We therefore have that the equilibrium ratios M/D and A/D satisfy the worst case collateral constraint (15) if and only if

$$h\left(\lambda^*\right) = 1 + \bar{\lambda}\left(1 - \ell\right). \tag{A.24}$$

Since the threshold shock depends on the interest rate i^F via (A.22), an elastic equilibrium exists if and only if

$$\left(h \circ f\right)\left(i^F\right) \leq 1 + \bar{\lambda}\left(1 - \ell\right). \tag{A.25}$$

We now show that there is an interbank interest rate i^{F*} such that the existence condition

(A.25) holds if and only if $i^F \geq i^{F*}$. Indeed, the function h is strictly decreasing: we have $h'(\lambda^*) = -(1-\rho_F\ell)(1-G(\lambda^*)) < 0$. Moreover, we have $h(-\bar{\lambda}) = 1 + (1-\rho_F\ell)\bar{\lambda}$ and $h(\bar{\lambda}) = 1$, the latter due to our assumption that the mean of $\tilde{\lambda}$ is zero. The composite function is strictly increasing in i^F with $(h \circ f)\left((1-\rho_F) i^S + \rho_F i^M\right) = 1$ and

$$(h \circ f)\left(i^S\right) = 1 + (1-\rho_F\ell)\bar{\lambda} > 1 + \bar{\lambda}(1-\ell)$$

It follows that there is a unique threshold funds rate $i^{F*} \in \left(i^M, i^S\right)$. that solves (A.25) with equality.

We have established the existence of an elastic equilibrium for all $i^F \geq i^{F*}$, and we have shown that there is no elastic equilibrium for lower values of i^F.

Step 2. Consider next inelastic equilibria such that the worst case leverage constraint (15) binds. We again begin by determining the equilibrium objects given a candidate interest rate i^F. The threshold shock λ^* satisfies (A.24). Moreover, the ratios M/D and A/D must satisfy the worst case leverage constraint (15) as well as reserve market clearing (24). Since the function h is decreasing with $h(\bar{\lambda}) = 1$, we have $\lambda^* \leq \bar{\lambda}$. We thus find a nonnegative M/D from reserve market clearing (24) and we can solve for a nonnegative A/D from the definition of the threshold (18) as long as (A.23) holds. Finally, the interest rate on other assets and the multiplier γ follow from (22).

To establish when an inelastic equilibrium exists, it remains to verify that $\gamma \geq 0$. Suppose we evaluate the first condition for reserves (22a) at λ^* defined by (A.24). By construction of the function f above, we have $\gamma \geq 0$ if and only if $\lambda^* \leq f(i^F)$. Indeed, the right hand side of (22a) is increasing in $G(\lambda^*$ and the cdf G is strictly increasing. Using the definition of the critical interest rate i^{F*} in (A.25), the condition $\lambda^* \leq f(i^F)$ is equivalent to $i^F \leq i^{F*}$.

We have shown that an inelastic equilibrium for all $i^F \geq i^{F*}$, and that there is no inelastic equilibrium for higher values of i^F.

Part (c) . Suppose now that $i^F = i^M$. The market clearing condition (24) holds as an inequality. Equilibrium is no longer unique. However, we show that for any candidate reserve ratio M/D, we can uniquely determine all other equilibrium objects, so the reserve ratio indexes equilibria. The first order condition for reserves (22a) can only hold if $\gamma > 0$ so the worst case collateral constraint (15) must hold with equality, and determines A/D given M/D. The threshold shock λ^* follows from (18). The interest rate i^A and the multiplier γ are again found from (22).

We need A/D to be nonnegative. This implies an upper bound for M/D is $\ell^{-1}\left(1 + \bar{\lambda}(1-\ell)\right)$. The lower bound follows from (24). To derive it, we first eliminate A/D from (15) and the definition of the liquidity threshold (18) to express the reserve-deposit ratio as

$$\frac{M}{D} = \frac{(1 - \rho_F \ell)\,\lambda^* + \bar{\lambda}\,(1 - \ell)}{(1 - \rho_F)\,\ell}. \tag{A.26}$$

Now substituting for M/D, we can rewrite market clearing (24) as

$$\int_{-\bar{\lambda}}^{\lambda^*} \left(\lambda^* - \tilde{\lambda}\right) dG\left(\tilde{\lambda}\right) \leq \lambda^* + \bar{\lambda}\frac{1 - \ell}{1 - \rho_F \ell} \tag{A.27}$$

We show that the inequality holds if λ^* is sufficiently large. We then have from (A.26) that it holds if M/D is sufficiently large

Consider the left hand side (LHS) and right hand side (RHS) of (A.27) as functions of λ^* on the interval $[-\bar{\lambda}, \bar{\lambda}]$. The LHS is strictly increasing with slope $G(\lambda^*) \leq 1$ and satisfies $LHS(-\bar{\lambda}) = 0$ as well as $LHS(\bar{\lambda}) = \bar{\lambda}$. The RHS is also strictly increasing with slope one and satisfies $RHS(-\bar{\lambda}) < 0)$ as well as $RHS(\bar{\lambda}) > \bar{\lambda}$. It follows that there is a unique value λ^{**} such that market clearing (24) holds if and only if and only if $\lambda^* \geq \lambda^{**}$.

It remains to verify that the implied lower bound for M/D is sufficiently small such that a nonnegative A/D can be determined from (15). We thus require

$$\ell\frac{M}{D} = \frac{(1 - \rho_F \ell)\,\lambda^{**} + \bar{\lambda}\,(1 - \ell)}{1 - \rho_F} \leq 1 + \bar{\lambda}\,(1 - \ell)\,,$$

which is satisfied as long as $\bar{\lambda}$ is sufficiently small since we know that $\lambda^{**} < \bar{\lambda}$.

Part (d). Suppose $i^F > i^M$. When the support bound $\bar{\lambda}$ of the liquidity shock distribution converges to zero, (24) implies that M/D, and hence in particular $\underline{m}$, also converges to zero. The limiting value for A/D then follows from the definition of λ^* in (18). If instead Suppose $i^F = i^M$, then the worst case collateral constraint (15) binds: taking the limit as $\bar{\lambda}$ goes to zero, we have $\ell M/D + \rho^A \ell A/D = 1$.■

A.3.4 Derivation of Equation (25

To derive the key linearized equation (25) for an elastic equilibrium with a corridor system, we start from (22). With a slack leverage constraint, we have $\gamma_t = 0$ and hence the simpler set of equations

$$
\begin{aligned}
i_t^S - i_t^F &= \left(i_t^F - i_t^M\right) \frac{\rho_F}{1-\rho_F} G\left(\lambda_t^*\right), \\
i_t^S - i_t^A &= \frac{\rho_A}{\rho_F}\left(i_t^S - i_t^F\right).
\end{aligned} \tag{A.28}
$$

Given the two policy rates, the five equations (A.28), (23) with $\gamma_t = 0$, (24) and the definition of the threshold shock

$$
\lambda_t^* = \frac{1 - \rho_F \ell \frac{M_t}{D_t} - \rho_A \ell \frac{A_t}{D_t}}{\left(1 - \rho_F \ell\right)}
$$

determine five variables: the balance sheet ratios M_t/D_t and A_t/D_t, the threshold λ_t^* and the interest rates on bank instruments i_t^A and i_t^D.

We loglinearize the definition of the threshold shock λ_t^* and the money market clearing condition:

$$
\lambda^* \hat{\lambda}_t^* = \frac{\rho_A \ell}{1-\rho_F \ell}\frac{A}{D}\left(\hat{d}_t - \hat{a}_t\right) + \frac{\rho_F \ell}{1-\rho_F \ell}\frac{M}{D}\left(\hat{d}_t - \hat{m}_t\right)
$$

$$
\frac{\ell\left(1-\rho_F\right)}{1-\rho_F \ell}\frac{M}{D}\left(\hat{d}_t - \hat{m}_t\right) = -G\left(\lambda^*\right)\lambda^* \hat{\lambda}_t^*
$$

We can substitute out the endogenous change in the ratio of reserves to deposits $\frac{M}{D}\left(\hat{d} - \hat{m}\right)$ to obtain

$$
\frac{1 - \rho_F + \rho_F G\left(\lambda^*\right)}{1-\rho_F}\lambda^* \hat{\lambda}_t^* = \frac{\rho_A \ell}{1-\rho_F \ell}\frac{A}{D}\left(\hat{d}_t - \hat{a}_t\right) \tag{A.29}
$$

Next, we loglinearize the first order condition for reserves – the first equation in (A.28) – to find

$$
\begin{aligned}
\frac{g(\lambda^\star)\lambda^\star}{G(\lambda^\star)}\hat{\lambda}_t^\star &= \frac{(1+r^F)(\delta - r^M)}{(\delta - r^F)(r^F - r^M)}\left(i_t^S - i_t^F\right) - \frac{1+r^M}{r^F - r^M}\left(i_t^S - i_t^M\right) \\
&= \left(\frac{(1+r^F)(\delta - r^M)}{(\delta - r^F)(r^F - r^M)} - \frac{1+r^M}{r^F - r^M}\right)\left(i_t^S - i_t^F\right) - \left(\frac{1+r^M}{r^F - r^M}\right)\left(i_t^F - i_t^M\right) \\
&= \frac{1+\delta}{\delta - r^F}\left(i_t^S - i_t^F\right) - \frac{1+r^M}{r^F - r^M}\left(i_t^F - i_t^M\right)
\end{aligned}
$$

Again assuming that net rates of return are small decimal numbers we obtain the approximation

$$
\frac{g\left(\lambda^*\right)\lambda^*}{G\left(\lambda^*\right)}\hat{\lambda}_t^* = \frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M}.
$$

Substituting for $\lambda^* \hat{\lambda}_t^*$ from (A.29) now leads to

$$\frac{g(\lambda^*)}{G(\lambda^*)} \frac{1-\rho_F}{1-\rho_F+\rho_F G(\lambda^*)} \frac{\rho_A \ell}{1-\rho_F \ell} \frac{A}{D} \left(\hat{d}_t - \hat{a}_t\right) = \frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M},$$

and rearranging delivers equation (25).

A.4 Characterization of steady state with banks

In this section, we prove Proposition 4.1.

A steady state equilibrium consists of a level of output, an inflation rate, a short rate, a quantity of real balances, and an equilibrium in fixed income markets.

Part (a). Fix an inflation rate π. Market clearing for money requires that demand for real balances from households equals supply from banks. In an equilibrium with a corridor system and small $\bar{\lambda}$, Proposition 4.2 (d) says that supply is pinned down by bank assets. The steady state market clearing condition analogous to (5) is therefore

$$\omega^\eta \left(\frac{\delta + \pi - i^D}{1+\delta+\pi}\right)^{-\eta} Y = \rho_A \ell A^r. \tag{A.30}$$

For any π, there is a unique deposit rate i^D that makes this equation hold.

To find policy parameters that generate this deposit rate, we substitute for the spread on other assets $i^S - i^A$ in (23) from (26), and use the fact that M/D is small to obtain that $\delta + \pi - i^D$ is proportional to $\delta + \pi - i^F$. It follows that there is a unique i^F consistent with money market clearing. Finally, we can choose i^M sufficiently low that this i^F is located above i^{F*}.

Part (b). In an equilibrium with a floor system, the supply of reserves is determined by the binding worst case leverage constraint (15) together with the government reserve supply rule (27). Market clearing for money means

$$\omega^\eta \left(\frac{\delta + \pi - i^D}{1+\delta+\pi}\right)^{-\eta} Y = \ell M^r + \rho^A \ell A^r. \tag{A.31}$$

For any π and M^r, there is a unique i^D that makes this equation hold.

In a floor system, (22) implies that spreads on all assets are proportional to the policy spread $i^S - i^M$. From (23), we then have that $\delta + \pi - i^D$ is proportional to $\delta + \pi - i^M$. It follows that there is a unique i^M consistent with money market clearing. ■